# CENTRES AND CENTRALIZERS OF IWAHORI-HECKE ALGEBRAS 

by

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#### Abstract

This thesis describes the minimal basis for the centre of an Iwahori-Hecke algebra.

Defining the Iwahori-Hecke algebra $\mathcal{H}$ over $\mathbb{Z}\left[\xi_{s}\right]_{s \in S}$ - a subring of the traditional ring of definition - we begin by defining positivity in $\mathcal{H}$, and by describing a very simple basis for the centralizer of the subalgebra generated by the element corresponding to a single simple reflection of the Coxeter group. This basis provides the building blocks for the results to follow on the centre.

The main result is that the set of primitive minimal positive elements of the centre forms a basis for the centre over $\mathbb{Z}\left[\xi_{s}\right]_{s \in S}$. The elements of this basis we call "class elements", and the basis itself the "minimal basis". It has many important properties aside from its minimality. For instance, it can be characterized as exactly those elements which specialize (on setting $\xi_{s}=0$ for all $s \in S$ ) to the sum of terms in a conjugacy class, and which have no other terms which are shortest elements of any other conjugacy class.

The approach is entirely independent of character theory, relying only on Coxeter group properties, making the whole theory very combinatorial in nature. This has several advantages, among which is that it can be generalized to centralizers of parabolic subalgebras, and we do this in some cases.

We define and make use of a constructive algorithm to explicitly find any class element. This algorithm relies on the basis of the centralizer of an element corresponding to a single generator, and reveals some further useful information about the class elements.

Finally we look at an important application of this work to the Brauer homomorphism for Iwahori-Hecke algebras defined by Jones.


## Introduction

This thesis began as an attempt to generalize the norm results of Lenny Jones [J2] over $\mathbb{Q}\left[q, q^{-1}\right]$ from type $A$ to type $B$ and other types, but has ended departing from norms and arriving with the concept of the minimal basis. We will however use norms at several key points in the thesis, and will return to them when we apply the results to the Jones definition of the Brauer homomorphism in type $A$.

The results here form part of the growing body of research developing the analogy between group algebras of Coxeter groups and their $q$-analogues, the IwahoriHecke algebras. So far however, much of this development has required fairly sophisticated techiniques, such as character theory, Kazhdan-Lusztig cell theory, and the Tits deformation theorem. A key point of this thesis is that the results here are purely combinatorial in nature, in that we only require results on the properties of finite Coxeter groups.

Specifically, of course, we are looking at the relationship between centres and centralizers of the group algebra of a Coxeter group $W$, and of an Iwahori-Hecke algebra. In the group algebra case, the centre of $\mathbb{Z} W$ has a well-known basis consisting of the sum of elements in a conjugacy class $C$, often denoted $\underline{C}$. This basis can be described using a characteristic function on elements of the conjugacy class. For instance, if $\chi_{C}(w)=\delta_{C C^{\prime}}$ where $w \in C^{\prime}$ and $\delta$ is the Kronecker delta, then $\sum_{w \in W} \chi_{C}(w) w=\underline{C}$. This characteristic function can be written as a linear combination of the irreducible characters of $W$.

In the $q$-analogue, the Iwahori-Hecke algebras, the characteristic function does not yield a basis for the centre. However it is possible to find a $q$-analogue of the characteristic function in terms of a linear combination of the irreducible characters of $\mathcal{H}$, which does the job. The possibility of this was pointed out to me in 1995 by John Graham, and in late 1996 when my work on the approach presented here was already well advanced, I received a pre-print of a paper by Meinolf Geck and Raphael Rouquier ([GR]) in which they do exactly this.

The approach which we develop here involves recognizing that the conjugacy class sums $\underline{C}$ in the group algebra are the minimal elements of a poset on the
"positive cone" of $Z(\mathbb{Z} W)$. The generalization of this statement to Iwahori-Hecke algebras forms the main theorem of the thesis: the primitive minimal positive central elements are an integral basis for the centre of $\mathcal{H}$, which we call the "minimal basis". The elements of this basis (the "class elements") are characterized by two facts: they specialize to a conjugacy class sum; and apart from that conjugacy class sum, there are no shortest elements of any conjugacy class in them. It will turn out that the Geck-Rouquier basis is in fact the minimal basis.

The proof of this is entirely combinatorial in nature, and relies only on properties of Coxeter groups, including the theorem of Geck and Pfeiffer. Thus one advantage our approach has is to remove the representation theory from the problem.

Another advantage is computational. We provide an algorithm for calculating elements of the minimal basis, starting from the sum of terms corresponding to shortest elements in a conjugacy class, and adding terms until a class element is reached. This algorithm is computationally more direct than using characters, as with the character theory approach, one needs to evaluate the image of the irreducible character on every basis element $\tilde{T}_{w}$ of $\mathcal{H}$. The algorithm we provide means only those with non-zero coefficient in the class element are calculated.

A third advantage is that our approach is thoroughly generalizable to centralizers of parabolic subalgebras, and to non-crystallographic Coxeter groups. In the group algebra case, the characteristic function for $J$-conjugacy classes (where $J$ is a subset of the set of simple reflections $S$ ) provides a basis of $J$-conjugacy class sums. But it is not possible to write this characteristic function in terms of irreducible characters when $J \neq S$. So an Iwahori-Hecke algebra analogy is not possible via character theory. Our methods are readily generalizable to parabolic subalgebras, subject to proving an analogy of the Geck-Pfeiffer theorem for $J$-conjugacy classes. We provide proofs of some specific cases throughout the thesis.

Chapter 1 begins with the standard definitions and results in Coxeter groups and Iwahori-Hecke algebras, including the result of Geck and Pfeiffer on conjugacy classes of Weyl groups.

Chapter 2 sets up some of the new structures used in the thesis. We define positivity in the Iwahori-Hecke algebra, and then introduce a partial order on the
positive part of the algebra, and the positive part of the centre. This sets up the statement of the main result for centres, which is that if $W$ is either a Weyl group, dihedral, or of type $H_{3}$, a certain subset of the minimal elements of the positive part of the centre of the corresponding Iwahori-Hecke algebra (the set of "primitive" minimal elements) is a $\mathbb{Z}\left[\xi_{s}\right]_{s \in S}$-basis for the centre. Further, there is a simple characterization of these primitive minimal elements.

Chapter 3 contains the proofs of the main result for centres as set up in Chapter
2. A core piece of this minimal structure is presented in the first section, with the basis for the centralizer of the generator $\tilde{T}_{s}$ for some simple reflection $s$. We show the existence of the class elements in section two. These are elements which specialize to the conjugacy class sum, and which contain no other shortest elements of other conjugacy classes. Their existence is necessary for the proof of the main result, and has been shown in the case of centres by Geck and Rouquier using character theory when $W$ is a Weyl group. The existence of such elements may also be derived from the results of Jones in type $A$, and a description of both of these existence proofs is found in $[\mathrm{Fr}]$ and Appendix A.

This chapter also contains the definition of the algorithm for constructing the class elements, the proof that these class elements are the primitive minimal positive central elements, and the proof that they are an integral basis for the centre.

Chapter 4 contains a generalization of the main result of chapter two to centralizers of certain parabolic subalgebras, using the chain theory introduced by Brieskorn and Saito in [BS]. Chapter 5 describes in detail several examples of the minimal basis, including those corresponding to dihedral groups, type $H_{3}$, and small groups of type $A_{2}, A_{3}, A_{4}, B_{2}$, and $B_{3}$.

Finally Chapter 6 applies the results on the minimal basis to the Brauer homomorphism for Iwahori-Hecke algebras. We begin by deriving several more results following from the algorithm, which yield important information about the class elements. This information is vital for the results on the Brauer homomorphism. We then describe the image of the minimal basis, and the kernel of the homomorphism in terms of the minimal basis. Chapter 7 does some combinatorics in finding a correspondence between $J$-conjugacy classes in a conjugacy class in type
$A_{n}$ and certain compositions of $n+1$.
Note: Since the submission of this thesis, I have found that the Geck-Pfeiffer result (Theorem (1.1.2)) has been proved for the non-crystallographic types as well. Thanks to Meinolf Geck for this information. The proofs in these cases can be found in the paper [GHLMP], and are done using the computer algebra system CHEVIE (a relative of GAP, which was used for the crytallographic cases).

This means that the results in this thesis stated excluding type $H_{4}$ are now valid for any finite Coxeter group. In particular, Theorem (2.2.1), Lemmas (3.2.3), (3.2.5), and Proposition (3.4.2) are now completely general. Also, Proposition (5.1.1) (for the dihedral groups) and (5.2.1) and (5.2.2) (for $H_{3}$ ) can be found in [GHLMP].

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This thesis was typeset using $\mathcal{A} \mathcal{M} \mathcal{S}-\mathrm{T}_{\mathrm{E}} \mathrm{X}$. The diagrams were produced using Paul Taylor's Commutative Diagrams package.

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## Index of Notation

| Q | the field of rational numbers. |
| :---: | :---: |
| $\mathbb{Z}$ | the ring of integers. |
| $\mathbb{N}$ | the set of non-negative integers (including zero). |
| $R$ | $\mathbb{Z}\left[\xi_{s}\right]_{s \in S}$ |
| $R^{+}$ | $\mathbb{N}\left[\xi_{s}\right]_{s \in S}$ |
| F | $\mathbb{Q}\left(\xi_{s}\right)_{s \in S}$ |
| W | a finite Coxeter group. |
| $S$ | the generating set for $W$ |
| $J$ | a subset of $S$ |
| $W_{J}$ | the parabolic subgroup of $W$ generated by $J$ |
| $\mathfrak{D}_{J, K}$ | the set of distinguished $W_{J}-W_{K}$ double coset representatives in $W$ |
| C | a conjugacy class of $W$ |
| $\mathfrak{C}$ | a $J$-conjugacy class of $W$ for $J \subseteq S$ |
| $c c l(W)$ | the set of conjugacy classes of $W$ |
| $\operatorname{ccl}_{J}(W)$ | the set of $J$-conjugacy classes of $W$ |
| $l_{\mathfrak{C}}$ | the length of the shortest elements of $\mathfrak{C}$ |
| $\mathfrak{C}_{\text {min }}$ | the set of shortest elements of $\mathfrak{C}$ |
| $\mathfrak{C}_{w}$ | the $J$-conjugacy class containing $w$ |
| $\mathfrak{C}^{w}$ | the equivalence class in $\mathfrak{C}$ containing $w$ generated by $\sim_{s}$ for $s \in J$ |
| $\mathcal{H}_{q}$ | the Iwahori-Hecke algebra over $\mathbb{Z}\left[q_{s}^{1 / 2}, q_{s}^{-1 / 2}\right]_{s \in S}$ |
| $\mathcal{H}$ | the Iwahori-Hecke algebra over $R$ |
| $\mathcal{H}_{J}$ | the Iwahori-Hecke algebra generated by $\left\{\tilde{T}_{s} \mid s \in J\right\}$ |
| $\mathcal{H}^{+}$ | the Iwahori-Hecke algebra over $R^{+}$ |
| $Z(\mathcal{H})$ | the centre of $\mathcal{H}$ |
| $Z(\mathcal{H})_{\text {min }}^{+}$ | the set of primitive minimal positive central elements of $\mathcal{H}$ |
| $Z_{\mathcal{H}}\left(\mathcal{H}_{J}\right)$ | the centralizer of $\mathcal{H}_{J}$ in $\mathcal{H}$ |

## Chapter 1 Preliminaries

We begin by setting out some basic definitions and general results we will need on finite Coxeter groups and Iwahori-Hecke algebras of finite Coxeter groups.

### 1.1. Coxeter Groups

Let $W$ be a finite Coxeter group with generating set $S$ of simple reflections. Then for $s, s^{\prime} \in S, W$ has relations

$$
\begin{array}{r}
s^{2}=1 \\
\left(s s^{\prime}\right)^{m_{s s^{\prime}}}=1
\end{array}
$$

for some $m_{s s^{\prime}} \in \mathbb{N}$. We refer to a word as an element of $W$ written as a product of the generators. In general there are many words in the elements of $S$ which represent the same group element $w \in W$. If $r \in \mathbb{N}$ is minimal such that $s_{1} s_{2} \ldots s_{r}=w$ for $s_{i} \in S$, then we say $s_{1} \ldots s_{r}$ is a reduced expression for $w$, and that the length of $w$ is $l(w)=r$. Finite Coxeter groups are classified according to their root systems (see for example $[\mathrm{H}]$ ), and those corresponding to a crystallographic root system are called Weyl groups. This thesis will deal exclusively with finite Coxeter groups.

If $J \subseteq S$, the parabolic subgroup $W_{J}$ of $W$ is the subgroup generated by the set of simple reflections $J$. Each parabolic subgroup of a Coxeter group is a direct product of Coxeter groups.

Each Coxeter group is partitioned into conjugacy classes $C$. We may also partition $W$ into $J$-conjugacy classes, corresponding to sets of elements conjugate by elements of $W_{J}$. These are sometimes called orbits of $W_{J}$ under conjugation. We denote the set of all $J$-conjugacy classes in $W$ by $c c l_{J}(W)$, and abbreviate $c c l_{S}(W)$ to $\operatorname{ccl}(W)$. To distinguish reference to conjugacy classes and $J$-conjugacy classes, we will use $C$ to denote an $(S$-) conjugacy class, and $\mathfrak{C}$ for a general $J$-conjugacy class when $J$ may be a subset of $S$.

For $w \in W$, we write $\mathfrak{C}_{w}$ for the $J$-conjugacy class containing $w$. Let $l_{\mathfrak{C}}$ be the length of the shortest elements in the $J$-conjugacy class $\mathfrak{C}$, and let $\mathfrak{C}_{\text {min }}$ be the set of shortest elements in $\mathfrak{C}$.

For any $J$-conjugacy class $\mathfrak{C}$ and $s \in J$ we can define an equivalence relation $\sim_{s}$ on $\mathfrak{C}$ by writing $w \sim_{s} u$ if $s w s=u$ and $l(w)=l(u)$. We then define the equivalence class $\sim_{J}$ to be generated by the relations $\sim_{s}$ for $s \in J$. The $\sim_{J}$-equivalence classes consist of elements of the same length which can be reached from each other by a sequence of conjugations by simple reflections from $J$, where each step in the sequence gives an element in $\mathfrak{C}$ of the same length.

Each $J$-conjugacy class $\mathfrak{C}$ is the disjoint union of such $\sim_{J}$-equivalence classes, so we can specify uniquely the $\sim_{J}$-equivalence class by choosing a representative from it. We will denote the $\sim_{J}$-equivalence class containing $w$ by $\mathfrak{C}^{w}$.

For $w, w^{\prime} \in \mathfrak{C}$ for some $J$-conjugacy class $\mathfrak{C}$, we say $w \rightarrow_{J} w^{\prime}$ if there exists a sequence $r_{1}, r_{2}, \ldots, r_{m}$ of elements of $J$ and a sequence $w_{0}, \ldots, w_{m}$ of elements of $\mathfrak{C}$ such that if $w_{0}=w$, and $w_{i}=r_{i} w_{i-1} r_{i}(1 \leq i \leq m)$ then $w_{m}=w^{\prime}$, and $l\left(w_{i}\right) \leq l\left(w_{i-1}\right)$ with $w_{i} \neq w_{i-1}$, for $1 \leq i \leq m$.
(1.1.1) Definition. Let $\mathfrak{C}$ be a $J$-conjugacy class of $W$. We say $\mathfrak{C}$ is reducible if for all $w \in \mathfrak{C}$ there exists a $v \in \mathfrak{C}_{\text {min }}$ such that $w \rightarrow_{J} v$. Each $W_{J}-W_{J}$ double coset in $W$ is partitioned by $J$-conjugacy classes, and if every $J$-conjugacy class in the double coset $W_{J} d W_{J}$ is $J$-reducible, we say that the double coset $W_{J} d W_{J}$ is reducible.

The following result is from [GP], Theorem 1.1.
(1.1.2) Theorem. (Geck-Pfeiffer) Every conjugacy class $C$ of a Weyl group $W$ is reducible.

Furthermore, if $w$ and $w^{\prime} \in C_{\min }$, then there exists a sequence of $x_{i} \in W$ and $w_{i} \in C_{\min }$ such that $w=w_{0}, x_{i} w_{i} x_{i}^{-1}=w_{i+1}$, and $w_{n}=w^{\prime}$, with either $l\left(x_{i} w_{i}\right)=l\left(x_{i}\right)+l\left(w_{i}\right)$ or $l\left(w_{i} x_{i}^{-1}\right)=l\left(w_{i}\right)+l\left(x_{i}^{-1}\right)$ for each $i$.

This theorem is an invaluable tool for the results in this thesis.
(1.1.3) Corollary. Let $w \in C \in \operatorname{ccl}(W)$, with $l(w)>l_{C}$. Then there exists a $u \in C^{w}$ and an $s \in S$ such that $l(s u s)=l(u)-2$.

This means that in every equivalence class $C^{w}$ not containing shortest elements from $C$, there is at least one element which shortens on conjugation by a simple
reflection.
(1.1.4) Remark. We can define a partial order on these equivalence classes as follows. We say $C^{w} \geq C^{u}$ if $l(w) \geq l(u)$ and there exists a $w^{\prime} \in C^{w}$ and $u^{\prime} \in C^{u}$ such that $w^{\prime} \rightarrow_{S} u^{\prime}$. If either of those inequalities are equalities, then both are, and we have $C^{w}=C^{u}$. That is, $u \in C^{w}$ and vice versa.

In this context, (1.1.3) tells us that for each $C^{w}$ with $l(w) \neq l_{C}$, there exists a set $C^{u}$ such that $C^{w}>C^{u}$, and (1.1.3) gives the existence of a $C^{v}$ for $v \in C_{\min }$ such that $C^{w}>C^{v}$.
(1.1.5) Example. The Weyl group of type $A_{3}$ is generated by $S=\left\{s_{1}, s_{2}, s_{3}\right\}$ with relations $s_{i}^{2}=\left(s_{1} s_{2}\right)^{3}=\left(s_{2} s_{3}\right)^{3}=\left(s_{1} s_{3}\right)^{2}=1$. The Hasse diagram for the above poset on the conjugacy class $C=\left\{s_{1} s_{2}, s_{2} s_{1}, s_{2} s_{3}, s_{3} s_{2}, s_{1} s_{2} s_{3} s_{2}, s_{2} s_{3} s_{2} s_{1}\right.$, $\left.s_{1} s_{2} s_{1} s_{3}, s_{3} s_{1} s_{2} s_{1}\right\}$ is:


### 1.2. Iwahori-Hecke Algebras

Let $\left\{q_{s} \mid s \in S\right\}$ be a set of indeterminates such that $q_{s}=q_{t}$ if $s$ and $t$ are conjugate in the finite Coxeter group $W$. We define the Iwahori-Hecke algebra $\mathcal{H}_{q}$ corresponding to $W$ to be the associative $\mathbb{Z}\left[q_{s}^{1 / 2}, q_{s}^{-1 / 2}\right]_{s \in S^{-}}$-algebra generated by the set $\left\{T_{s}\right\}_{s \in S}$, with relations

$$
\begin{align*}
T_{s}^{2} & =q_{s} T_{1}+\left(q_{s}-1\right) T_{s},  \tag{*}\\
\underbrace{\tilde{T}_{s} \tilde{T}_{s^{\prime}} \tilde{T}_{s} \ldots}_{m_{s s^{\prime}}} & =\underbrace{\tilde{T}_{s^{\prime}} \tilde{T}_{s} \tilde{T}_{s^{\prime}} \ldots}_{m_{s s^{\prime}}}
\end{align*}
$$

If $w=s_{1} \ldots s_{i}$ is a reduced expression for $w$, then we define

$$
T_{w}:=T_{s_{1}} \ldots T_{s_{i}} .
$$

As well as being an algebra generated by $\left\{T_{s}\right\}_{s \in S}, \mathcal{H}_{q}$ is then a $\mathbb{Z}\left[q_{s}^{1 / 2}, q_{s}^{-1 / 2}\right]_{s \in S^{-}}$ module with basis $\left\{T_{w}\right\}_{w \in W}$.

Let $R=\mathbb{Z}\left[\xi_{s}\right]_{s \in S}$, where $\xi_{s}=q_{s}^{1 / 2}-q_{s}^{-1 / 2}$ for each $s \in S$. Then $R$ is a subring of $\mathbb{Z}\left[q^{1 / 2}, q^{-1 / 2}\right]_{s \in S}$. Let $R^{+}=\mathbb{N}\left[\xi_{s}\right]_{s \in S}, R_{0}=\mathbb{Z}$, and $R_{0}^{+}=\mathbb{N}$, where we adopt the convention that $\mathbb{N}$ includes zero. Further, we write $F=\mathbb{Q}\left(\xi_{s}\right)_{s \in S}$ for the field of fractions of $R$.

We will find it useful to change the base ring of $\mathcal{H}_{q}$ from $\mathbb{Z}\left[q_{s}^{1 / 2}, q_{s}^{-1 / 2}\right]_{s \in S}$ to the subring $R$ by setting $\tilde{T}_{s}=q_{s}^{-1 / 2} T_{s}$, to give us an $R$-subalgebra denoted $\mathcal{H}$. The relation $(*)$ in $\mathcal{H}$ then becomes

$$
\tilde{T}_{s}^{2}=\tilde{T}_{1}+\xi_{s} \tilde{T}_{s}
$$

which has the obvious benefit of being simpler. It also has valuable positivity properties, since for any two basis elements $\tilde{T}_{x}$ and $\tilde{T}_{y} \in \mathcal{H}$, their product $\tilde{T}_{x} \tilde{T}_{y}=$ $\sum_{w \in W} f_{x, y, w} \tilde{T}_{w}$ has all coefficients $f_{x, y, w}$ in $R^{+}=\mathbb{N}\left[\xi_{s}\right]_{s \in S}$. Thus the product of any two elements of $\mathcal{H}$ whose coefficients are from $R^{+}$(that is, they are linear combinations of the $\tilde{T}_{w}$ over $R^{+}$) also has coefficients in $R^{+}$. These observations motivate the definition of $\mathcal{H}^{+}$in the next chapter.

If $X$ is a subset of $W$ (for example a conjugacy class), we denote by $\tilde{T}_{X}$ the following sum:

$$
\tilde{T}_{X}:=\sum_{x \in X} \tilde{T}_{x}
$$

The algebra $\mathcal{H}_{q}$ can be obtained from $\mathcal{H}$ by the following change of coefficient ring:

$$
\mathcal{H}_{q} \cong \mathbb{Z}\left[q_{s}^{1 / 2}, q_{s}^{-1 / 2}\right]_{s \in S} \otimes_{R} \mathcal{H}
$$

Consequently the centre of $\mathcal{H}$ embeds in the centre of $\mathcal{H}_{q}$, and an $R$-basis of $Z(\mathcal{H})$ will become a $\mathbb{Z}\left[q_{s}^{1 / 2}, q_{s}^{-1 / 2}\right]_{s \in S^{-}}$-basis for $Z\left(\mathcal{H}_{q}\right)$ after changing $\xi_{s}$ back to $q_{s}^{1 / 2}-q_{s}^{-1 / 2}$ and $\tilde{T}_{s}$ back to $q_{s}^{-\frac{1}{2}} T_{s}$. We deal with this, and with obtaining a $\mathbb{Z}\left[q_{s}, q_{s}^{-1}\right]_{s \in S}$-basis from the minimal $R$-basis, at the end of section 3.3.

Frequent use will be made of specializing the parameters $\xi_{s}$ to zero (equivalent to setting $q_{s}=1$ in $\mathcal{H}_{q}$ ), so for any $h \in \mathcal{H}$, we write $h_{0}=\left.h\right|_{\xi_{s}=0}$ for this specialization.

The following is proven in [J2: (2.4)], using properties of Frobenius algebras. We will give a combinatorial proof in section 3.1.
(1.2.1) Lemma. Let $w \in C_{\min }$ for $C \in \operatorname{ccl}(W)$. Then the element

$$
N_{W, 1}\left(\tilde{T}_{w}\right)=\sum_{u \in W} \tilde{T}_{u^{-1}} \tilde{T}_{w} \tilde{T}_{u}
$$

is in the centre $Z(\mathcal{H})$.
(1.2.2) Lemma. The set $\left\{N_{W, 1}\left(\tilde{T}_{w_{C}}\right) \mid C \in \operatorname{ccl}(W), w_{C} \in C_{\min }\right\}$ is linearly independent over $F$, and is contained in $Z(\mathcal{H})$.

Proof. The centrality follows from (1.2.1).
Write $N_{C}:=N_{W, 1}\left(\tilde{T}_{w_{C}}\right)$, and suppose there is some relation $\sum_{C} f_{C} N_{C}=0$ for $f_{C} \in F$. We will show that this implies that there is a non-trivial relationship among conjugacy class sums.

Firstly, we may assume that all $f_{C} \in R$, by multiplying out denominators. We may also assume that the set $\left\{f_{C}\right\}_{C}$ has no common factor of $\xi_{s}$ for any $s \in S$. Choose an arbitrary $s^{\prime} \in S$. Then there must be some $f_{C}$ such that $\left.f_{C}\right|_{\xi_{s^{\prime}}=0} \neq 0$, and so $\left.\left.\sum_{C} f_{C}\right|_{\xi_{s^{\prime}}=0} N_{C}\right|_{s_{s^{\prime}}=0} \neq 0$ since none of the $N_{C}$ will specialize to zero (they each have $a_{C} \tilde{T}_{C} \leq N_{C}$ for $\left.a_{C} \in \mathbb{N}\right)$. That is, $\left.\left.\sum_{C} f_{C}\right|_{\xi_{s^{\prime}}=0} N_{C}\right|_{\xi_{s^{\prime}}=0}=0$ is a non-trivial relation.

We now may remove all common factors of elements of $\left\{\xi_{s}\right\}_{s \in S \backslash\left\{s^{\prime}\right\}}$ from the set $\left\{\left.f_{C}\right|_{\xi_{s^{\prime}}=0}\right\}$. Then by choosing another $s \in S \backslash\left\{s^{\prime}\right\}$, we may continue in this fashion, eventually obtaining (after specializing each $\xi_{s}$ to zero) our desired relation

$$
\left.\sum_{C} f_{C}^{\prime} N_{C}\right|_{\xi_{s}=0, s \in S}=0
$$

with not all $f_{C}^{\prime} \in \mathbb{N}$ equal to zero. This is a contradiction, since $\left.N_{C}\right|_{\xi_{s}=0, s \in S}=$ $a_{C} \tilde{T}_{C}$ for some $a_{C} \in \mathbb{N}$, so the above relation is a relation between conjugacy class sums, which are linearly independent.

In fact the set of norms as stated in (1.2.2) is an $F$-basis for $Z\left(\mathcal{H}_{F}\right)$. This fact may be easily obtained using the Tits deformation theorem. However, the above (more elementary) lemma will suffice for our purposes.

## Chapter 2 <br> The main results

### 2.1 Positivity

We now make a brief excursion into higher generality to define positivity and obtain some basic consequences of the definition.
 with basis $X=\left\{x_{1}, \ldots, x_{n}\right\}$. Let $A_{0}$ be the free $\mathbb{Z}$-module with the same basis. Then $A=\mathbb{Z}\left[\xi_{i}\right]_{i \in I} X$ and $A_{0}=\mathbb{Z} X$.

Let $\operatorname{Mon}_{j}:=\operatorname{Mon}_{j}\left(\xi_{i}\right)_{i \in I}$ be the set of monomials in $\left\{\xi_{i} \mid i \in I\right\}$ of order $j$. That is, the set of products of the $\xi_{i}$ whose exponent sum is $j$. We will make extensive use of this set of monomials, and so to reduce repetition, we will use the notation $\mathfrak{m}_{j}$ to denote an arbitrary element of $\mathrm{Mon}_{j}$.

We may consider $A$ as a $\mathbb{Z}$-module with basis $\left\{\mathfrak{m}_{j} x_{k} \mid \mathfrak{m}_{j} \in \operatorname{Mon}_{j}, j \geq 0,1 \leq\right.$ $k \leq n\}$. Then we have

$$
A=\sum_{\substack{j \geq 0 \\ \mathfrak{m}_{j} \in \text { Mon }_{j}}} A_{0} \mathfrak{m}_{j} .
$$

It would be natural to consider $\mathbb{N}\left[\xi_{i}\right]_{i \in I}$ to be the positive part of the ring $\mathbb{Z}\left[\xi_{i}\right]_{i \in I}$, and there is a similarly natural partial order on the elements of $\mathbb{N}\left[\xi_{i}\right]_{i \in I}$ : if $f=\sum_{j \geq 0, \mathfrak{m}_{j} \in \operatorname{Mon}_{j}} f_{\mathfrak{m}_{j}} \mathfrak{m}_{j}$ and $g=\sum_{j \geq 0, \mathfrak{m}_{j} \in \operatorname{Mon}_{j}} g_{\mathfrak{m}_{j}} \mathfrak{m}_{j}$ for $f_{\mathfrak{m}_{j}}, g_{\mathfrak{m}_{j}} \in \mathbb{N}$, then $f \leq g$ if and only if $f_{\mathfrak{m}_{j}} \leq g_{\mathfrak{m}_{j}}$ for all $j$. An equivalent expression of this is to say $f \leq g$ if and only if $g-f \in \mathbb{N}\left[\xi_{i}\right]_{i \in I}$.

This partial order on the positive part of $\mathbb{Z}\left[\xi_{i}\right]_{i \in I}$ induces a partial order on the positive part of any free $\mathbb{Z}\left[\xi_{i}\right]_{i \in I}$-module. Since $A=\mathbb{Z}\left[\xi_{i}\right]_{i \in I} X$, define the positive part of $A$ to be $A^{+}=\mathbb{N}\left[\xi_{i}\right]_{i \in I} X$. Define a partial order on $A^{+}$by saying that if $a=\sum_{1 \leq k \leq n} a_{k} x_{k}$ and $b=\sum_{1 \leq k \leq n} b_{k} x_{k}$ where $a_{k}, b_{k} \in \mathbb{N}\left[\xi_{i}\right]_{i \in I}$, then $a \leq b$ in $A^{+}$if and only if $a_{k} \leq b_{k}$ in $\mathbb{N}\left[\xi_{i}\right]_{i \in I}$ for all $x_{k} \in X$. This is equivalent to saying $a \leq b$ in $A^{+}$if and only if $b-a \in A^{+}$.

There is an equally obvious partial order on $A_{0}^{+}=\mathbb{N} X$. If $a_{0}=\sum_{x_{k} \in X} c_{k} x_{k}$ and $b_{0}=\sum_{x_{k} \in X} d_{k} x_{k}$ for $c_{k}, d_{k} \in \mathbb{N}$ then $a_{0} \leq b_{0}$ in $A_{0}$ if and only if $c_{k} \leq d_{k}$ in $\mathbb{N}$ for $1 \leq k \leq n$, which is equivalent to having $b_{0}-a_{0} \in A_{0}^{+}$.

If we were to turn $A$ into a $\mathbb{Z}\left[\xi_{i}\right]_{i \in I}$-algebra by defining a multiplication between elements of its basis $X$ such that $x_{l} x_{m} \in \sum_{k=1}^{n} \mathbb{N}\left[\xi_{i}\right]_{i \in I} x_{k}$, then we have the following self-evident lemma:
(2.1.1) Lemma. If $x_{l} x_{m} \in \sum_{k=1}^{n} \mathbb{N}\left[\xi_{i}\right]_{i \in I} x_{k}$ for all $1 \leq l, m \leq n$, then sums and products of elements of $A^{+}$are also in $A^{+}$.

This may seem a severe restriction on the multiplication, but as pointed out in section 1.2, Iwahori-Hecke algebras satisfy this property.

Although we don't use the definition of intersection or union in this thesis, they seem potentially useful, so we give them here.

Define the intersection $x \cap y$ of elements $x, y \in \mathbb{N} X=A_{0}^{+}$to be the maximal element $z \in \mathbb{N} X$ such that $z \leq x$ and $z \leq y$. For any $y, z \in \mathbb{N} X$, we may write $y=\sum_{1 \leq k \leq n} y_{k} x_{k}$, and $z=\sum_{1 \leq k \leq n} z_{k} x_{k}$, for $y_{k}, z_{k} \in \mathbb{N}$. Then

$$
y \cap z=\sum_{1 \leq k \leq n} \min \left(y_{k}, z_{k}\right) x_{k} .
$$

Similarly we define the union $y \cup z$ of elements $y, z \in \mathbb{N} X$ to be the minimal element in $\mathbb{N} X$ greater than both $y$ and $z$. Then for the above $y$ and $z$ we have

$$
y \cup z=\sum_{1 \leq k \leq n} \max \left(y_{k}, z_{k}\right) x_{k} .
$$

We can define an analogous intersection and union for any two positive elements of the $\mathbb{Z}\left[\xi_{i}\right]_{i \in I^{-}}$algebra $A$. Let $a, b \in A^{+}$. Then $a \cap b$ is the maximal element $c \in A^{+}$such that $c \leq a$ and $c \leq b$. For any $a, b \in A^{+}$, we may write $a=\sum_{0 \leq j \leq N, \mathfrak{m}_{j} \in \operatorname{Mon}_{j}} a_{\mathfrak{m}_{j}} \mathfrak{m}_{j}$ and $b=\sum_{0 \leq j \leq M, \mathfrak{m}_{j} \in \operatorname{Mon}_{j}} b_{\mathfrak{m}_{j}} \mathfrak{m}_{j}$, for $a_{\mathfrak{m}_{j}}, b_{\mathfrak{m}_{j}} \in \mathbb{N} X$. Then

$$
a \cap b=\sum_{\substack{0 \leq j \leq \min (M, N) \\ \mathfrak{m}_{j} \in \operatorname{Mon}_{j}}}\left(a_{\mathfrak{m}_{j}} \cap b_{\mathfrak{m}_{j}}\right) \mathfrak{m}_{j} .
$$

Similarly the union $a \cup b$ of $a, b \in A^{+}$is the minimal element greater than both $a$ and $b$. With $a$ and $b$ as defined above we have

$$
a \cup b=\sum_{\substack{\left.0 \leq j \leq \max (M, N) \\ \mathfrak{m}_{j} \in \operatorname{Mon}_{j}\right)}}\left(a_{\mathfrak{m}_{j}} \cup b_{\mathfrak{m}_{j}}\right) \mathfrak{m}_{j}
$$

### 2.2 The main Results

 elements of the partial ordering $\left(B^{+}, \leq\right)$, and similarly let $\min \left(B_{0}^{+}\right)$be the set of non-zero minimal elements of the poset $\left(B_{0}^{+}, \leq_{0}\right)$.

The elements of $\min \left(A_{0}^{+}\right)$are simply the elements of $X$ (which are a $\mathbb{Z}$-basis for $\left.A_{0}\right)$, and the elements of $\min \left(A^{+}\right)$are $\mathfrak{m}_{j}$-multiples of elements of $X$ for $\mathfrak{m}_{j} \in \operatorname{Mon}_{j}$ (which are a $\mathbb{Z}$-basis for $A$ ). The sets $\min \left(A^{+}\right)$and $\min \left(B^{+}\right)$are not finite if $A$ and $B$ are non-trivial. For example if $a \neq 0$ is minimal in $B^{+}$, then so is $\xi_{i} a$, and so is $\xi_{i}^{2} a$, and so on.

We will restrict attention to a set of representatives of $\min \left(B^{+}\right)$(so as to exclude $\mathfrak{m}_{j}$-multiples). Write

$$
a=\sum_{\substack{j \geq 0 \\ \mathfrak{m}_{j} \in \operatorname{Mon}_{j}}} a_{\mathfrak{m}_{j}} \mathfrak{m}_{j} \in B^{+}
$$

with $a_{\mathfrak{m}_{j}} \in \mathbb{N} X$. We will call $a$ primitive if the set of monomials $\mathfrak{m}_{j}$ in $a$ with $a_{\mathfrak{m}_{j}} \neq 0$ has no common factor.

Let $B_{\min }^{+}$be the set of primitive minimal elements of the poset $\left(B^{+}, \leq\right)$. We then have

$$
\min \left(B^{+}\right)=\bigcup_{\substack{j \geq 0 \\ \mathfrak{m}_{j} \in \text { Mon }_{j}}} \mathfrak{m}_{j} B_{\min }^{+}
$$

In this thesis we look at $A=\mathcal{H}$, indexing set $I=S$ (the set of simple reflections of $W$ ), with $R$-basis $\left\{\tilde{T}_{w} \mid w \in W\right\}$, and $B=Z(\mathcal{H})$. The multiplication between the elements $\tilde{T}_{w} \in \mathcal{H}^{+}$has the positivity property required for lemma (2.1.1), so the conclusion holds: that the sums and products of elements of $\mathcal{H}^{+}$are also in $\mathcal{H}^{+}$. This is a simple yet significant benefit of moving to the ring $R$.

For the group algebra $R W$, we have that primitive minimal positive elements of the centre $Z(R W)$ are conjugacy class sums, and so form an $R$-basis of $Z(R W)$. The analogous result for the Iwahori-Hecke algebra would be that $Z(\mathcal{H})_{\min }^{+}$is an $R$-basis for $Z(\mathcal{H})$, and this is our main result below. Another aspect of the analogy is that all elements of $Z(R W)_{\min }^{+}$have non-zero specialization. This carries over to $\mathcal{H}$ also.

Our main theorem for centres is then the following.
(2.2.1) Theorem. Let $W$ be any finite Coxeter group except that of type $H_{4}$. Then
(i) $Z(\mathcal{H})_{\min }^{+}$is an $R$-basis for $Z(\mathcal{H})$,
(ii) $h \in Z(\mathcal{H})_{\min }^{+}$if and only if
a) $h_{0}=\tilde{T}_{C}$ for some conjugacy class $C$ of $W$, and
b) $h-\tilde{T}_{C}$ contains no shortest elements of any conjugacy class of $W$.

The proof of the Weyl group part of (2.2.1) is contained in section 3.2. This theorem is proved using only properties of the Coxeter groups, such as the GeckPfeiffer theorem, which we needed to generalize to dihedral groups and $H_{3}\left(H_{4}\right.$ is excluded from the above theorem for computational reasons relating to the generalization of (1.1.2)).

Thus we have an elementary approach to the centres of Iwahori-Hecke algebras, including a description of the $R$-basis not only in terms of minimal elements of an easily defined poset, but a characterization of these minimal elements which is entirely analogous to the classical case. This characterization indicates that in fact the basis found by Geck and Rouquier in [GR] is the same as the minimal basis, as their elements satisfy properties a) and b) above.

Our approach to the proof of this theorem will be to build up from a minimal basis for the centralizer of a single generator, $\tilde{T}_{s}$. Such a basis is found in section 3.1. This approach lends itself naturally to generalization to arbitrary centralizers, and we pursue this in several cases later in the thesis. This kind of generalization is not possible using the character theory approaches such as used in $[\mathrm{R}],[\mathrm{C} 2]$ and [GR].

We will also provide an algorithm for computing the minimal basis, which is elementary, and which is analogous to an algorithm for computing the elements of a conjugacy class when one is given the shortest elements. This we do in section 3.3. The algorithm serves not only as a computational tool, but the fact that it is well defined leads to further details about the basis which becomes very useful in our applications to the Brauer homomorphism in chapter six.

We also will generalize (2.2.1) to centralizers of certain parabolic subalgebras. To do this, it is necessary (and sufficient, in fact), to generalize (1.1.2) to $J$ -
conjugacy classes. We do this for several cases in chapters four and five, giving us:
(2.2.2) Theorem. Let $J \subseteq S$, and suppose that either: $J$ is principal in types $A$ and $B$; or $|J|=1$ or 2 , and $W$ is a Weyl group. Then
(i) $Z_{\mathcal{H}}\left(\mathcal{H}_{J}\right)_{\min }^{+}$is an $R$-basis for $Z_{\mathcal{H}}\left(\mathcal{H}_{J}\right)$,
(ii) $h \in Z_{\mathcal{H}}\left(\mathcal{H}_{J}\right)_{\min }^{+}$if and only if
a) $h_{0}=\tilde{T}_{\mathfrak{C}}$ for some $J$-conjugacy class $\mathfrak{C}$ of $W$, and
b) $h-\tilde{T}_{\mathfrak{C}}$ contains no shortest elements of any J-conjugacy class of $W$.

## Chapter 3 Centres of Iwahori-Hecke algebras

### 3.1 THE $s$-CLASS ELEMENTS

We now introduce the fundamental building blocks of any central element. Indeed, the building blocks of any element which commutes with an element of a parabolic subalgebra of $\mathcal{H}$. These are the basis elements of the centralizer of a single generator $\tilde{T}_{s}$, which we will call $s$-class elements. Considering the centralizer of such a small subalgebra is a natural viewpoint, since the centralizer of any parabolic subalgebra is contained in the centralizer of each generator of the parabolic subalgebra. So in particular, the centre is simply the intersection of the set of centralizers of all generators:

$$
Z(\mathcal{H})=\bigcap_{s \in S} Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)
$$

If $J=\{s\} \subseteq S$, we will call a $J$-conjugacy class an $s$-conjugacy class. Every $s$-conjugacy class is contained in a double $\operatorname{coset}\langle s\rangle d\langle s\rangle$, for some $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}$, the set of distinguished $\langle s\rangle-\langle s\rangle$ double coset representatives in $W$. The double cosets may be classified according to whether the intersection $\langle s\rangle^{d} \cap\langle s\rangle$ is 1 or $\langle s\rangle$, and this provides the following means of listing all the possible types of $s$-conjugacy class.

If the intersection is 1 , then $d s \neq s d$, and otherwise $d s=s d$. So every double coset either consists of elements $\{d, d s, s d, s d s\}$ in the trivial intersection case, or $\{d, d s\}$ in the non-trivial intersection case. We can then list the $s$-conjugacy classes as follows: if $d s \neq s d$, we have $\{d, s d s\}$ and $\{s d, d s\}$; if $d s=s d$ we have $\{d\}$ and $\{d s\}$.

The basis for the centralizer of $s$ in $\mathbb{Z} W=R_{0} W$ is the set of $s$-conjugacy class sums. From the above this is the set of all elements of form $d$ or $d s$ if $d s=s d$, and all elements of form $d+s d s$ or $d s+s d$ if $d s \neq s d$. Furthermore, these are the minimal elements of $\left(Z_{R_{0} W}(s), \leq_{0}\right)$. We will give the Iwahori-Hecke algebra analogy in (3.1.9).

The following lemma is inspired by (2.4) in [DJ2].
(3.1.1) Lemma. Let $c=\sum_{w \in W} r_{w} \tilde{T}_{w}$ for $r_{w} \in R$, and let $s \in S$. Then $c$ is in $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$ if and only if for all d distinguished in $\langle s\rangle d\langle s\rangle$ such that $s d \neq d s$ we have
(i) $r_{d s}=r_{s d}$, and
(ii) $r_{s d s}=r_{d}+\xi_{s} r_{d s}$.

Proof. Firstly $c$ is in the centralizer if and only if the sum of terms from each $\langle s\rangle-\langle s\rangle$-double coset commutes with $\tilde{T}_{s}$.

Given any $\langle s\rangle$ - $\langle s\rangle$-double coset with $d$ distinguished, if $d s=s d$ then the double coset consists of the elements $d$ and $d s$, and each corresponding element $\tilde{T}_{d}$ and $\tilde{T}_{d s}$ commutes with $\tilde{T}_{s}$. Thus the sum $r_{d} \tilde{T}_{d}+r_{d s} \tilde{T}_{d s}$ commutes with $\tilde{T}_{s}$ for any $r_{d}, r_{d s} \in R$.

If $d s \neq s d$, then the double coset sum is $r_{d} \tilde{T}_{d}+r_{d s} \tilde{T}_{d s}+r_{s d} \tilde{T}_{s d}+r_{s d s} \tilde{T}_{s d s}$. This commutes with $\tilde{T}_{s}$ if and only if

$$
\tilde{T}_{s}\left(r_{d} \tilde{T}_{d}+r_{d s} \tilde{T}_{d s}+r_{s d} \tilde{T}_{s d}+r_{s d s} \tilde{T}_{s d s}\right)=\left(r_{d} \tilde{T}_{d}+r_{d s} \tilde{T}_{d s}+r_{s d} \tilde{T}_{s d}+r_{s d s} \tilde{T}_{s d s}\right) \tilde{T}_{s} .
$$

The left hand side is

$$
\begin{aligned}
& r_{d} \tilde{T}_{s d}+r_{d s} \tilde{T}_{s d s}+r_{s d}\left(\tilde{T}_{d}+\xi_{s} \tilde{T}_{s d}\right)+r_{s d s}\left(\tilde{T}_{d s}+\xi_{s} \tilde{T}_{s d s}\right)= \\
& \quad r_{s d} \tilde{T}_{d}+r_{s d s} \tilde{T}_{d s}+\left(r_{d}+\xi_{s} r_{s d}\right) \tilde{T}_{s d}+\left(r_{d s}+\xi_{s} r_{s d s}\right) \tilde{T}_{s d s}
\end{aligned}
$$

and the right hand side is

$$
\begin{aligned}
& r_{d} \tilde{T}_{d s}+r_{d s}\left(\tilde{T}_{d}+\xi_{s} \tilde{T}_{d s}\right)+r_{s d} \tilde{T}_{s d s}+r_{s d s}\left(\tilde{T}_{s d}+\xi_{s} \tilde{T}_{s d s}\right)= \\
& r_{d s} \tilde{T}_{d}+\left(r_{d}+\xi_{s} r_{d s}\right) \tilde{T}_{d s}+r_{s d s} \tilde{T}_{s d}+\left(r_{s d}+\xi_{s} r_{s d s}\right) \tilde{T}_{s d s}
\end{aligned}
$$

Equating coefficients of $\tilde{T}_{d}$ or $\tilde{T}_{s d s}$ gives (i), and of $\tilde{T}_{d s}$ or $\tilde{T}_{s d}$ gives (ii).
This lemma has some direct and useful consequences for elements of the centre, and in particular the positive part of the centre.

## (3.1.2) Corollary.

If $h=\sum_{w \in W} r_{w} \tilde{T}_{w} \in Z(\mathcal{H})$ and $u, u^{\prime} \in C^{u}$, then $r_{u}=r_{u^{\prime}}$.
If in addition $h \in Z(\mathcal{H})^{+}$, then for $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}, d \notin Z_{W}(s)$ and $r \in R^{+}$we have:
(i) $r \tilde{T}_{d} \leq h \Longrightarrow r\left(\tilde{T}_{d}+\tilde{T}_{s d s}\right) \leq h$,
(ii) $r \tilde{T}_{d s}$ or $r \tilde{T}_{s d} \leq h \Longrightarrow r\left(\tilde{T}_{d s}+\tilde{T}_{s d}+\xi_{s} \tilde{T}_{s d s}\right) \leq h$,
(iii) $\tilde{T}_{s d s} \leq h \Longrightarrow \tilde{T}_{d}+\tilde{T}_{s d s} \leq h$.
(iv) $\tilde{T}_{w} \leq h \Longrightarrow \tilde{T}_{C} \leq h$ where $C$ is the conjugacy class containing $w$.

Proof. The first statement, and parts (i) and (ii) of the second statement are direct consequences of the lemma. Parts (iii) and (iv) follow since the coefficient is one, and we are dealing with positive elements. That is, if $r_{s d s}=1$, then $1=r_{d}+\xi_{s} r_{d s}$, which implies $r_{d} \leq 1$ and $\xi_{s} r_{d s} \leq 1$. The latter is impossible by definition, unless $r_{d s}=0$, giving $r_{d}=1$. This proves (iii), and (iv) is a generalization of this.
(3.1.3) Proposition. Suppose $w \in W$ is not minimal in its conjugacy class, and $h \in Z(\mathcal{H})$. Then the coefficient of $\tilde{T}_{w}$ in $h$ is an $R^{+}$-linear combination of coefficients in $R$ of strictly shorter elements in $h$. In fact, it is an $R^{+}$-linear combination of the coefficients in $R$ of shortest elements of conjugacy classes.

Proof. The first statement follows from the first statement of (3.1.2), (1.1.3), and (3.1.1)(ii). The second follows by induction.
(3.1.4) Corollary. If there exists an element $h \in Z(\mathcal{H})$, such that
(i) $h_{0}=a \tilde{T}_{C}$ for some $a \in \mathbb{Z}$, and
(ii) there are no shortest elements from any conjugacy class in $h-h_{0}$, then $h$ is the unique central element with these properties.

Proof. Suppose $h^{\prime} \in Z(\mathcal{H})$ has the property that $h_{0}^{\prime}=a \tilde{T}_{C}$ and $h^{\prime}-h_{0}^{\prime}$ has no shortest elements from any conjugacy class. Then $h^{\prime}-h \in Z(\mathcal{H})$ has no shortest elements of any conjugacy class. Thus by (3.1.3), $h^{\prime}-h=0$.

We return to the centralizer of $\tilde{T}_{s}$ in the Iwahori-Hecke algebra. The minimal $R_{0}$-basis for $Z_{R_{0} W}(s)$ is the set of $s$-conjugacy class sums, as noted at the start of this section. We now provide the analogy in $\mathcal{H}$.
(3.1.5) Definition. Let $d$ be distinguished in $\langle s\rangle d\langle s\rangle$. We define the following four types of elements, and call them s-class elements because they correspond to
$s$-conjugacy classes:

$$
\begin{array}{ll}
\text { Type I, } d \in Z_{W}(s): \quad b_{d}^{I}=\tilde{T}_{d} \\
& b_{d s}^{I}=\tilde{T}_{d s} \\
\text { Type II, } d \notin Z_{W}(s): \quad b_{d}^{I I}=\tilde{T}_{d}+\tilde{T}_{s d s} \\
& b_{d s}^{I I}=\tilde{T}_{d s}+\tilde{T}_{s d}+\xi_{s} \tilde{T}_{s d s}
\end{array}
$$

Note that every distinguished $\langle s\rangle$ - $\langle s\rangle$-double coset representative either commutes with $s$ or it doesn't. Also note that when $\xi_{s}=0$, these elements correspond to sums of $s$-conjugate elements, and each element of an $\langle s\rangle$ - $\langle s\rangle$-double coset appears with coefficient one in exactly one $s$-class element.

Later we will use diagrams to represent the structure of central elements, and the core "cells" of these diagrams will be those corresponding to $s$-class elements. The type II $s$-class elements may be represented graphically by the following diagrams:


Figure (3.1.6)
(3.1.7) Proposition. Let $s \in S$. The set of $s$-class elements $\left\{b_{d}^{I}, b_{d s}^{I}, b_{d}^{I I}, b_{d s}^{I I} \mid d \in\right.$ $\left.\mathfrak{D}_{\langle s\rangle,\langle s\rangle}\right\}$ is an $R$-basis for $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$.

Proof. On specialization to $\xi_{s}=0$, each $s$-class element becomes a sum of $s$ conjugate elements in the group algebra. Such sums are a basis for the centralizer of $s$ in the group algebra, and in particular are linearly independent. It follows that the $s$-class elements are also linearly independent. It is also easy to check that each $s$-class element is in the centralizer of $\tilde{T}_{s}$.

Let $h$ be an element of $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$, and write $r_{w} \in R$ for the coefficient of $\tilde{T}_{w}$ in $h$. Then, as in the proof for (3.1.1), we may write $h$ as an $R$-linear combination
of sums of terms corresponding to elements in an $\langle s\rangle-\langle s\rangle$ double coset. If the distinguished representative $d$ of the double coset is in the centralizer $Z_{W}(s)$, then $r_{d} \tilde{T}_{d}+r_{d s} \tilde{T}_{d s}=r_{d} b_{d}^{I}+r_{d s} b_{d s}^{I}$ - a linear combination of $s$-class elements - so we need only to check the case when $d \notin Z_{W}(s)$. Using the relations from (3.1.1), we have the following:

$$
\begin{aligned}
r_{d} \tilde{T}_{d}+r_{d s} \tilde{T}_{d s}+r_{s d} \tilde{T}_{s d}+r_{s d s} \tilde{T}_{s d s} & =r_{d} \tilde{T}_{d}+r_{d s} \tilde{T}_{d s}+r_{d s} \tilde{T}_{s d}+\left(r_{d}+\xi_{s} r_{d s}\right) \tilde{T}_{s d s} \\
& =r_{d}\left(\tilde{T}_{d}+\tilde{T}_{s d s}\right)+r_{d s}\left(\tilde{T}_{d s}+\tilde{T}_{s d}+\xi_{s} \tilde{T}_{s d s}\right) \\
& =r_{d} b_{d}^{I I}+r_{d s} b_{d s}^{I I}
\end{aligned}
$$

which is a linear combination of $s$-class elements. Thus we have that any $h \in$ $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$ may be written
where $r_{w}$ is the coefficient of $\tilde{T}_{w}$ in $h$. Thus $h$ is a linear combination of $s$-class elements, and spanning follows.
(3.1.8) Corollary. Let $\mathfrak{D}_{\langle s\rangle,\langle s\rangle}$ be the set of distinguished $\langle s\rangle-\langle s\rangle$-double coset representatives in $W$. Then rank $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)=2\left|\mathfrak{D}_{\langle s\rangle,\langle s\rangle}\right|$.

Proof. For each double coset, there are two distinct basis elements for $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$.
An element of the centre is also an element of any centralizer in $\mathcal{H}$, and in particular the centralizers of elements $\tilde{T}_{s}$ for each $s \in S$. Using the same principle as in Figure (3.1.6), it may thus be represented as a graph with terms of form $\mathfrak{m}_{k} \tilde{T}_{w}$ with $\mathfrak{m}_{k} \in \operatorname{Mon}_{k}$ as nodes, and lines labelled by simple reflections connecting each node with the other terms in its $s$-class element for each $s \in S$. This provides a graphical way to check if an element is in the centre: ensure that for each $s \in S$ every node is part of a unique $s$-class element sub-graph.
(3.1.9) Lemma. The set of all s-class elements is the set $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)_{\min }^{+}$.

Proof. The $s$-class elements are all clearly primitive and minimal, so the converse needs to be established.

Let $h \in Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)^{+}$be non-zero. We will show that $h$ is greater than or equal to some $s$-class element.

By (3.1.7), $h$ is an $R$-linear combination of the $s$-class elements. It suffices to show that it is in fact an $R^{+}$-linear combination. Since $h$ is non-zero, there must be some $s$-class element $b$ with non-zero coefficient $r \in R$. But if $w$ is a shortest element of the $s$-conjugacy class corresponding to $b$, then $w$ only occurs in the $s$-class element $b$ (and with coefficient one in $b$ ), meaning that $\tilde{T}_{w}$ has coefficient $r$ in $h$. But since $h$ is in $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)^{+}$, the coefficient of every term $\tilde{T}_{u}$ for $u \in W$ is positive, and so $r$ must be positive, proving the lemma.

Proof of (1.2.1). To show $N=N_{W, 1}\left(\tilde{T}_{w}\right)$ is in the centre for $w \in C_{\min }$ for some conjugacy class $C$, we need to show $N$ is in every centralizer $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$ for $s \in S$.

Consider the left cosets $\langle s\rangle d$ of $\langle s\rangle$ in $W$. Each left coset has two elements, $d$ and $s d$, where $d$ is distinguished, and $W$ is partitioned by these left cosets. Let $\mathfrak{D}_{s}$ be the set of distinguished left coset representatives of $\langle s\rangle$ in $W$. Then we may write

$$
\begin{aligned}
N & =\sum_{u \in W} \tilde{T}_{u} \tilde{T}_{w} \tilde{T}_{u^{-1}} \\
& =\sum_{d \in \mathfrak{D}_{s}}\left(\tilde{T}_{d} \tilde{T}_{w} \tilde{T}_{d^{-1}}+\tilde{T}_{s d} \tilde{T}_{w} \tilde{T}_{d^{-1} s}\right) \\
& =\sum_{d \in \mathfrak{D}_{s}}\left(\tilde{T}_{d} \tilde{T}_{w} \tilde{T}_{d^{-1}}+\tilde{T}_{s} \tilde{T}_{d} \tilde{T}_{w} \tilde{T}_{d^{-1}} \tilde{T}_{s}\right)
\end{aligned}
$$

Each product $\tilde{T}_{d} \tilde{T}_{w} \tilde{T}_{d^{-1}}$ is an $R$-linear combination of terms $\tilde{T}_{x}$ for $x \in W$, and so $N$ is an $R$-linear combination of terms of form $\tilde{T}_{x}+\tilde{T}_{s} \tilde{T}_{x} \tilde{T}_{s}$, for $x \in W$. It now suffices to check that for every $x \in W$, the sum $\tilde{T}_{x}+\tilde{T}_{s} \tilde{T}_{x} \tilde{T}_{s}$ is in $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$. This is an elementary task, and can be checked by going through the possibilities for $x$ is the double coset $\langle s\rangle x\langle s\rangle$, as we have done earlier in this section.

### 3.2 The existence of the class elements

There are several ways to prove the existence of class elements - the elements which will make up the minimal basis in 3.4 (definition given below). In [Fr], the author used the existence of certain elements which had been found in type $A_{n}$ by Jones in [J2], and other elements found for general Weyl groups by Geck
and Rouquier in [GR]. We include those methods in Appendix A. However we provide here an independent existence proof, which is more combinatorial. The Jones methods used relative norms - a technique more difficult in types other than $A_{n}$, as for example the conjugacy classes in $B_{n}$ do not correspond so nicely with parabolic subgroups. The Geck-Rouquier methods rely on character theory, an approach which does not carry over to subalgebras.

The techniques in this section simply adopt the important characteristics of the characters which are needed in [Fr] to obtain the upper bound, and generalizes these to other functions satisfying these properties. This allows us potentially to apply them to centralizers as well as the centre case.

We want to find elements of $Z(\mathcal{H})$ which are analogous to conjugacy class sums in the group algebra. The following elements will turn out to fill that role.
(3.2.1) Definition. If $\Gamma_{C} \in Z(\mathcal{H})^{+}$, then $\Gamma_{C}$ is called a class element if it satisfies the following two properties:
(3.2.1.1) $\left.\Gamma_{C}\right|_{\xi_{s}=0, s \in S}=\tilde{T}_{C}$, and
(3.2.1.2) $\Gamma_{C}-\tilde{T}_{C}$ contains no terms of shortest length in any conjugacy class.

The purpose of this section is to prove such elements exist. This in fact is already known when $W$ is a Weyl group, as mentioned several times above, through the work of Geck and Rouquier [GR], however the methods we pursue are more general.

For some fixed $h=\sum_{w \in W} r_{w} \tilde{T}_{w} \in Z(\mathcal{H})$, define the function $\mathfrak{h}: \mathcal{H} \rightarrow R$ by setting $\mathfrak{h}\left(\tilde{T}_{w}\right)=r_{w}$ and extending linearly to the whole of $\mathcal{H}$.
(3.2.2) Lemma. For all $w, v \in W, \mathfrak{h}\left(\tilde{T}_{w} \tilde{T}_{v}\right)=\mathfrak{h}\left(\tilde{T}_{v} \tilde{T}_{w}\right)$.

Proof. We first prove for all $w \in W$ when $l(v)=1$, setting $v=s \in S$. It suffices to consider $w \in\langle s\rangle d\langle s\rangle$ for some $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}$. Suppose firstly that $d s=s d$. Then $w=d$ or $d s$, and in fact we have $\tilde{T}_{d} \tilde{T}_{s}=\tilde{T}_{s} \tilde{T}_{d}$ and $\tilde{T}_{d s} \tilde{T}_{s}=\tilde{T}_{s} \tilde{T}_{d s}$, so the lemma follows trivially. So we may suppose $d s \neq s d$, and thus $w=d, d s, s d$, or $s d s$. Then
using (3.1.1) we have

$$
\begin{aligned}
\mathfrak{h}\left(\tilde{T}_{d} \tilde{T}_{s}\right) & =\mathfrak{h}\left(\tilde{T}_{d s}\right)=r_{d s}=r_{s d}=\mathfrak{h}\left(\tilde{T}_{s} \tilde{T}_{d}\right) \\
\mathfrak{h}\left(\tilde{T}_{d s} \tilde{T}_{s}\right) & =\mathfrak{h}\left(\tilde{T}_{d}+\xi_{s} \tilde{T}_{d s}\right)=\mathfrak{h}\left(\tilde{T}_{d}\right)+\xi_{s} \mathfrak{h}\left(\tilde{T}_{d s}\right)=r_{d}+\xi_{s} r_{d s}=r_{s d s}=\mathfrak{h}\left(\tilde{T}_{s} \tilde{T}_{d s}\right) \\
\mathfrak{h}\left(\tilde{T}_{s d s} \tilde{T}_{s}\right) & =\mathfrak{h}\left(\tilde{T}_{s d}+\xi_{s} \tilde{T}_{s d s}\right)=r_{s d}+\xi_{s} r_{s d s}=r_{d s}+\xi_{s} r_{s d s}=\mathfrak{h}\left(\tilde{T}_{s} \tilde{T}_{s d s}\right) .
\end{aligned}
$$

The case $w=s d$ is exactly symmetric to the $w=d s$ case above.
Now suppose the lemma holds for all $w \in W$ when $l(v) \leq k$, and suppose $x \in W$ has length $l(x)=k+1$. Then $x=v s$ for some $v$ of length $k$ and some $s \in S$, that is, $l(x)=l(v)+l(s)$, and $\tilde{T}_{x}=\tilde{T}_{v} \tilde{T}_{s}$. Let $w \in W$. Then $\mathfrak{h}\left(\tilde{T}_{w} \tilde{T}_{x}\right)=\mathfrak{h}\left(\tilde{T}_{w} \tilde{T}_{v} \tilde{T}_{s}\right)=\mathfrak{h}\left(\tilde{T}_{s} \tilde{T}_{w} \tilde{T}_{v}\right)=\mathfrak{h}\left(\tilde{T}_{v} \tilde{T}_{s} \tilde{T}_{w}\right)=\mathfrak{h}\left(\tilde{T}_{x} \tilde{T}_{w}\right)$, with the second and third equalities following since $\tilde{T}_{w} \tilde{T}_{v}$ (resp. $\tilde{T}_{s} \tilde{T}_{w}$ ) are linear combinations of elements $\tilde{T}_{u} \in \mathcal{H}$, and $\mathfrak{h}$ is linear. So by induction we may pass $\tilde{T}_{s}$ and $\tilde{T}_{v}$ respectively through terms in the products $\tilde{T}_{w} \tilde{T}_{v}$ and $\tilde{T}_{s} \tilde{T}_{w}$ respectively. This proves the lemma.

Remark. Note that here we could define $\mathfrak{h}_{J}$ to correspond to a centralizer of a parabolic subalgebra as follows. For $h=\sum_{w \in W} \tilde{T}_{w} \in Z_{\mathcal{H}}\left(\mathcal{H}_{J}\right)$ we define the function $\mathfrak{h}_{J}: \mathcal{H} \rightarrow R$ setting $\mathfrak{h}_{J}\left(\tilde{T}_{w}\right)=r_{w}$. Then the above lemma would read "for all $w \in W$, and $v \in W_{J}, \mathfrak{h}_{J}\left(\tilde{T}_{w} \tilde{T}_{v}\right)=\mathfrak{h}_{J}\left(\tilde{T}_{v} \tilde{T}_{w}\right)$ ", and could be used in analogous results for the remainder of this section.

It is well known that the terms corresponding to shortest elements of a conjugacy class in a central element have the same coefficient. This has been proved by Ram $[\mathrm{R}]$ and Starkey [C2] in type $A$, and by Geck and Pfeiffer [GP] for general Weyl groups. However all these results used character theory, and part of our goal is to introduce a setup generalizable to centralizers, which characters are not. Since the function $\mathfrak{h}$ is generalizable, it is worth following through this approach.
(3.2.3) Lemma. Let $W$ be a Weyl group, and let $C$ be a conjugacy class of $W$, with $w, w^{\prime} \in C_{\min }$. Then if $h=\sum_{w \in W} r_{w} \tilde{T}_{w} \in Z(\mathcal{H})$ we have $r_{w}=r_{w^{\prime}}$.

Proof. By (1.1.2), we have the existence of a sequence of $x_{i} \in W$ and $w_{i} \in C_{\text {min }}$ such that $w=w_{0}, w_{n}=w^{\prime}, x_{i} w_{i} x_{i}^{-1}=w_{i+1}$ and either $l\left(x_{i} w_{i}\right)=l\left(x_{i}\right)+l\left(w_{i}\right)$ or $l\left(w_{i} x_{i}^{-1}\right)=l\left(w_{i}\right)+l\left(x_{i}^{-1}\right)$ for all $1 \leq i \leq n-1$. We may suppose without loss
of generality that $n=1$ and that there exists an $x \in W$ such that $x w x^{-1}=w^{\prime}$ and $l(x w)=l(x)+l(w)$. Note that this also implies $l\left(w^{\prime} x\right)=l\left(w^{\prime}\right)+l(x)$ since $x w=w^{\prime} x$ and $l(w)=l\left(w^{\prime}\right)$.

It follows that $\tilde{T}_{x} \tilde{T}_{w} \tilde{T}_{x}^{-1}=\tilde{T}_{w^{\prime}}$, since $\tilde{T}_{x} \tilde{T}_{w}=\tilde{T}_{x w}=\tilde{T}_{w^{\prime} x}=\tilde{T}_{w^{\prime}} \tilde{T}_{x}$. Thus

$$
\mathfrak{h}\left(\tilde{T}_{w}\right)=\mathfrak{h}\left(\tilde{T}_{x} \tilde{T}_{w} \tilde{T}_{x}^{-1}\right)=\mathfrak{h}\left(\tilde{T}_{w^{\prime}}\right)
$$

with the first equality by (3.2.2), and the lemma follows.
By (1.2.2), there exists a set of linearly independent elements of size $|\operatorname{ccl}(W)|$ in the centre of $\mathcal{H}$. By (3.2.3), the coefficients of the shortest elements of a conjugacy class are the same in any central element, so for any $h_{i} \in Z\left(\mathcal{H}_{F}\right)$ we may write

$$
\begin{equation*}
h_{i}=\sum_{j=1}^{r} a_{i, j} \tilde{T}_{C_{j, \min }}+X_{i} \tag{*}
\end{equation*}
$$

where $X_{i} \in \mathcal{H}_{F}$ contains no shortest elements of any conjugacy class with non-zero coefficient, and $a_{i, j} \in F$.

Let $\left\{h_{i}=\sum_{j=1}^{r} a_{i, j} \tilde{T}_{C_{j, \text { min }}}+X_{i} \in Z\left(\mathcal{H}_{F}\right) \mid C_{j} \in \operatorname{ccl}(W)\right\}$ be a set of linearly independent central elements in $\mathcal{H}_{F}$.
(3.2.4) Lemma. Let $h_{i}$ be as above. Then the set $\left\{h_{i}: 1 \leq i \leq r\right\}$ is linearly independent if and only if the set of vectors $\left\{\mathbf{a}_{i}=\left(a_{i, 1}, \ldots, a_{i, r}\right) \mid 1 \leq i \leq r\right\}$ is linearly independent.

Proof. Suppose that the $\mathbf{a}_{i}$ are not linearly independent, and there is some relation $\sum_{i} r_{i} \mathbf{a}_{i}=0$ for some $r_{i} \in F$. Then $\sum_{i} r_{i} a_{i, j}=0$ for all $j$, and we have

$$
\begin{aligned}
\sum_{i} r_{i} h_{i} & =\sum_{i} r_{i} \mathbf{a}_{i} \cdot\left(\tilde{T}_{C_{1, \text { min }}}, \ldots, \tilde{T}_{C_{r, \text { min }}}\right)+\sum_{i} r_{i} X_{i} \\
& =\sum_{i} r_{i} X_{i} .
\end{aligned}
$$

The left hand side of the equation is in the centre, so the right hand side is also. This is a contradiction by (3.1.3), since $\sum_{i} r_{i} X_{i}$ has no shortest elements with non-zero coefficient.

If on the other hand the $h_{i}$ are linearly dependent, we have $\sum_{i} r_{i} h_{i}=0$ for some $r_{i} \in F$, so

$$
\sum_{i} r_{i} \mathbf{a}_{i} \cdot\left(\tilde{T}_{C_{1, \min }}, \ldots, \tilde{T}_{C_{r, \min }}\right)+\sum_{i} r_{i} X_{i}=0
$$

and again since $X_{i}$ contains no shortest elements we may equate coefficients of shortest elements in $\tilde{T}_{C_{j, \min }}$ to give $\sum_{i} r_{i} a_{i, j}=0$ for each $j$, so $\sum_{i} r_{i} \mathbf{a}_{i}=0$ and the $\mathbf{a}_{i}$ are linearly dependent.

The following lemma may also be deduced from Geck and Rouquier's work in [GR], as shown in Appendix A.
(3.2.5) Lemma. For each conjugacy class $C$ in a Weyl group $W$ there exists an element in the centre $Z\left(\mathcal{H}_{F}\right)$ which contains shortest elements from $C$ with coefficient 1, and no other shortest elements from any conjugacy class.

Proof. As pointed out above there exist $r$ linearly independent elements $\left\{h_{i} \mid 1 \leq\right.$ $i \leq r\}$ in the centre of $\mathcal{H}_{F}$, and we can decompose them as in $(*)$. We can then write the vector equation

$$
\left(\begin{array}{c}
h_{1} \\
\vdots \\
h_{r}
\end{array}\right)=\left(\begin{array}{ccc}
a_{1,1} & \ldots & a_{1, r} \\
\vdots & & \vdots \\
a_{r, 1} & \ldots & a_{r, r}
\end{array}\right)\left(\begin{array}{c}
\tilde{T}_{C_{1, \text { min }}} \\
\vdots \\
\tilde{T}_{C_{r, \text { min }}}
\end{array}\right)+\left(\begin{array}{c}
X_{1} \\
\vdots \\
X_{r}
\end{array}\right)
$$

where the $a_{i, j}$ are in $F$, and the $X_{i}$ contain no shortest elements of any conjugacy class.

By Lemma (3.2.4), the rows of the matrix $A=\left(a_{i, j}\right)$ are linearly independent, so $A$ is invertible, and we have

$$
A^{-1}\left(\begin{array}{c}
h_{1} \\
\vdots \\
h_{r}
\end{array}\right)=\left(\begin{array}{c}
\tilde{T}_{C_{1, \min }} \\
\vdots \\
\tilde{T}_{C_{r, \min }}
\end{array}\right)+A^{-1}\left(\begin{array}{c}
X_{1} \\
\vdots \\
X_{r}
\end{array}\right) .
$$

Each entry of the vector on the left hand side is in the centre, and so on the right hand side we also have a vector whose entries are central elements. But each of these on the right hand side is $\tilde{T}_{C_{i, \text { min }}}$ plus a linear combination of elements $X_{j}$ of $\mathcal{H}_{F}$, none of whom contain any shortest elements of any conjugacy class.
(3.2.6) Theorem. For each conjugacy class $C$, there exists a class element $\Gamma_{C} \in$ $Z(\mathcal{H})^{+}$. That is, $\left.\Gamma_{C}\right|_{\xi_{s}=0}=\tilde{T}_{C}$ and, $\Gamma_{C}-\tilde{T}_{C}$ contains no shortest elements of any conjugacy class. Furthermore, $\Gamma_{C}$ is the unique element of $Z(\mathcal{H})^{+}$satisfying (3.2.1.1) and (3.2.1.2).

Proof. Recall from (3.1.3) that the coefficient of any element $\tilde{T}_{w}$ in a central element $h$ may be written as an $R^{+}$-linear combination of the coefficients of shortest
elements in $h$. From (3.2.5), we have the existence of an element $h_{C}=\tilde{T}_{C_{\text {min }}}+Y \in$ $Z\left(\mathcal{H}_{F}\right)$ where $Y$ contains no shortest elements of any conjugacy class. Since the only shortest elements of any conjugacy class in $h_{C}$ have coefficient 1, (3.1.3) implies every $\tilde{T}_{w}$ occuring in $h_{C}$ has coefficient in $R^{+}$, so $h_{C} \in Z(\mathcal{H})^{+}$.

Suppose $a \tilde{T}_{w} \leq h_{C}$ with $a$ an integer. That is, $h_{C}$ has non-zero specialization. Then by (3.1.2)(iv), we have $a \tilde{T}_{C^{\prime}} \leq h_{C}$ where $C^{\prime}$ is the conjugacy class containing $w$. So the shortest elements of $C^{\prime}$ appear in $h_{C}$, which means $C=C^{\prime}$ since $h_{C}$ contains shortest elements of only one conjugacy class, $C$. Further, the only shortest elements in $h_{C}$ have coefficient one, so $a=1$. Thus $a \tilde{T}_{w}=\tilde{T}_{w} \leq \tilde{T}_{C}$, and so the only terms with non-zero specialization in $h_{C}$ are from $\tilde{T}_{C}$, giving $h_{C}=\tilde{T}_{C}+X$, with $X$ specializing to zero.

For uniqueness, suppose there exists a $\Gamma_{C}^{\prime} \in Z(\mathcal{H})^{+}$satisfying (3.2.1.1) and (3.2.1.2). Then $\Gamma_{C}-\Gamma_{C}^{\prime}$ has no shortest elements of any conjugacy class with non-zero coefficient. This contradicts (3.1.3) unless $\Gamma_{C}=\Gamma_{C}^{\prime}$.

### 3.3 Constructing central elements - Algorithms

Having shown the existence of the class elements, we now seek to describe how to obtain them. This section will draw heavily on the $s$-class element basis for the centralizer $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$ found in 3.1. There are two algorithms we present - a "relative" one and an "absolute" one, the difference being that the former needs to refer to an upper bound. Under certain conditions, these will be shown to be the same, giving us an effective absolute algorithm for calculating class elements, independent of upper bounds.

Both these algorithms take an element of $\mathcal{H}^{+}$, and add terms to it until it is in the centre. The key property that will emerge is that when one starts with the sum of shortest elements in a conjugacy class, every addition is "same-length" or "longer" (in a sense that will become clear). This is a very powerful property, and we use it to draw further conclusions about the terms in the class elements in section 6.1, when we look at the Brauer homomorphism.

Given the $s$-class element basis for the centralizer of $\tilde{T}_{s}$ in $\mathcal{H}$ for any $s$, we may write a central element as a linear combination of $s$-class elements for any
$s \in S$. Indeed, for any element $h \in \mathcal{H}^{+}$and $s \in S$ we may write $h$ as a linear combination $h_{s}$ of $s$-class elements, plus a linear combination $h_{s}^{\prime}$ of terms $\mathfrak{m}_{k} \tilde{T}_{w}$ which are neither $R^{+}$-multiples of complete $s$-class elements (of type I) on their own, nor can they be summed with any other terms in $h_{s}^{\prime}$ to create an $R^{+}$-multiple of an $s$-class element. In other words, $h_{s}$ is a maximal linear combination of $s$-class elements less than or equal to $h$.
[Note that we do not claim that $h_{s}$ is the unique maximal linear combination of $s$-class elements less than $h$. This is not possible in general, as for example we could have $\xi_{s} \mathfrak{m}_{k} \tilde{T}_{s d s}+\xi_{s} \mathfrak{m}_{k} \tilde{T}_{d}+\mathfrak{m}_{k} \tilde{T}_{s d}+\mathfrak{m}_{k} \tilde{T}_{d s} \leq h$ for some $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}$ and $\mathfrak{m}_{k} \in \operatorname{Mon}_{k}$, and then either $\xi_{s} \mathfrak{m}_{k}\left(\tilde{T}_{d}+\tilde{T}_{s d s}\right)$ or $\mathfrak{m}_{k}\left(\tilde{T}_{d s}+\tilde{T}_{s d}+\xi_{s} \tilde{T}_{s d s}\right)$ are $s$-class elements which could be put into $h_{s}$.]

Having decomposed $h \in \mathcal{H}^{+}$in such a manner into $h=h_{s}+h_{s}^{\prime}$, we may then add terms to complete the $s$-class elements containing the terms in $h_{s}^{\prime}$. That is, we may add terms to $h$ to create a new element (say $\bar{h} \geq h$ ) which is a linear combination of $s$-class elements. In other words, $\bar{h}$ is a minimal element of $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$ greater than or equal to $h$. (Note again we do not claim uniqueness for such a minimal centralizer element greater than $h$ ). For all terms in $h_{s}^{\prime}$ not of form $\mathfrak{m}_{k} \tilde{T}_{s d s}$ (for $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}, \mathfrak{m}_{k} \in \operatorname{Mon}_{k}$ ) the added terms will in fact be unique, but the case of $\mathfrak{m}_{k} \tilde{T}_{s d s}$ with $k \geq 1$ could be considered either as part of the $s$-class element $b_{d}^{I I}$ or $b_{d s}^{I I}$.

This describes the nucleus of an algorithm for constructing a positive central element containing a given positive element. We could continue to construct elements of $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$ corresponding to different $s \in S$ by adding more and more terms until we (hopefully) eventually create an element in the centralizer of all $\tilde{T}_{s}$ for $s \in S$ - that is, in $Z(\mathcal{H})$. To ensure the algorithm stops, however (and does not continue to add elements ad infinitum), we need to either: ensure that terms of form $\mathfrak{m}_{k} \tilde{T}_{s d s}$ for $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}$ never occur in $h_{s}^{\prime}$ for any $s$ in $S$ and for any stage in the process (or define the algorithm more closely to ensure this); or provide an upper bound in $Z(\mathcal{H})$ which controls the additions to $h$ - and this we will further look at in definition (3.3.1) below.

For any $h \in \mathcal{H}^{+}$we can find a positive central element $c$ greater than $h$ (for
example, $h \leq N_{W, 1}(h) \in Z(\mathcal{H})^{+}$- see (1.2.1) or [J2] for a definition). Then we can apply the latter approach to ensure we can always construct a central element greater than or equal to $h$ and less than or equal to $c$, no matter in what order of additions we proceed. If we need to complete an $s$-class element for which $\xi_{s} \mathfrak{m}_{k} \tilde{T}_{s d s}$ is in $h$, we need to choose whether to consider it part of $b_{d}^{I I}$ or $b_{d s}^{I I}$ - in other words, whether to add $\xi_{s} \mathfrak{m}_{k} \tilde{T}_{d}$ or $\mathfrak{m}_{k}\left(\tilde{T}_{s d}+\tilde{T}_{d s}\right)$. We can decide this on the basis of which is less than $c-h \in \mathcal{H}^{+}$. Then our new element will remain less than or equal to $c$. If both options (either $\xi_{s} \mathfrak{m}_{k} \tilde{T}_{d}$ or $\left.\mathfrak{m}_{k}\left(\tilde{T}_{d s}+\tilde{T}_{s d}\right)\right)$ are less than $c-h$, the choice can be arbitrary. Terms other than $\xi_{s} \mathfrak{m}_{k} \tilde{T}_{s d s}$ are in uniquely defined $s$-class elements, and so a choice will never need to be made (see Lemma (3.1.2)(i) and (ii)).

Suppose $h=h_{s}+h_{s}^{\prime} \in \mathcal{H}^{+}$, with $h_{s}$ a maximal element of $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$ less than $h$. Let $m_{s}$ be the length of the shortest term in $h_{s}^{\prime}$ (for non-zero $h_{s}^{\prime}$ ). We now formalize the above with the following definition:
(3.3.1) Definition. Let $h \in \mathcal{H}^{+}$, with $h \leq c$ for some $c \in Z(\mathcal{H})^{+}$. Define the algorithm $\mathfrak{B}_{c}$ to conduct the following sequence of procedures:
(i) split $h$ into $h=h_{s}+h_{s}^{\prime}$ for each $s \in S$ with $h_{s}$ maximal in $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$ less than or equal to $h$;
(ii) if $h_{s}^{\prime}=0$ for all $s \in S$, stop;
$(i i)^{\prime}$ otherwise, evaluate $m_{s}$ for each $s$, and choose $s \in S$ such that $m_{s}$ is minimal;
(iii) add terms from $c-h$ which complete the $s$-class elements containing terms in $h_{s}^{\prime}$ of length $m_{s}$.
(iv) declare the new element to be $\mathfrak{B}_{c}(h)$, and repeat from (i) with new element.

Note that here we do not make $h$ into an element of a centralizer of $\tilde{T}_{s}$ for some $s$ immediately. We find the shortest term in $h$ which is not in a complete $s$-class element for some $s \in S$, and add the necessary terms to make that particular $s$-class element complete. The purpose of this aspect of the definition is that later we will use induction on the length $m_{s}$ of the shortest term in an incomplete $s$-class element. This will in fact allow us to ensure (by controlling $h$ and the element $c$ )
that we never complete the $s$-class element of a term of form $r \tilde{T}_{s d s}$ in $h_{s}^{\prime}$ for some $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}$.

An immediate consequence of the definition is the following.
(3.3.2) Lemma. Let $h \leq c \in Z(\mathcal{H})^{+}$. Suppose that after $n$ iterations of $\mathfrak{B}_{c}$ the shortest term of $\mathfrak{B}_{c}^{n}(h)$ in an incomplete s-class element for some $s \in S$ has length $m_{s}$ in $\mathfrak{B}_{c}^{n}(h)$. Then every element of length $<m_{s}$ in $\mathfrak{B}_{c}^{n}(h)$ is in a complete $s$-class element, for all $s \in S$.

Our main aim is to construct class elements - elements of the centre which specialize to the conjugacy class sum $\tilde{T}_{C}$, and which contain no other shortest elements of any conjugacy class. The existence of such elements we have shown in section 3.2. We now show that using the class elements $\Gamma_{C}$ as upper bounds, we can start the algorithm on $\tilde{T}_{C}$ and will never need to add shorter elements. With this, every step of the algorithm becomes uniquely defined, and it is never necessary to make a choice relative to the upper bound. In other words, we will show paradoxically that given the existence of certain types of element which we can use as an upper bound, upper bounds are not necessary when starting with $\tilde{T}_{C}$.
(3.3.3) Proposition. The algorithm $\mathfrak{B}_{\Gamma_{C}}^{i}\left(\tilde{T}_{C}\right)$ never needs to refer to the upper bound $\Gamma_{C}$ for any $i \in \mathbb{N}$.

Proof. There is only need to refer to the upper bound if it is necessary at some point to decide whether to consider an element of form $\xi_{s} \mathfrak{m}_{k} \tilde{T}_{s d s}$ (for some $d \in$ $\left.\mathfrak{D}_{\langle s\rangle,\langle s\rangle}\right)$ as part of the $s$-class element $b_{d}^{I I}$ or $b_{d s}^{I I}$. That is, we will need to refer to the upper bound if it is necessary at some point to add a shorter element or elements. We claim that under the conditions of the proposition it is never necessary to add shorter at any point in the construction, and we prove this by induction on the number of repeats of $\mathfrak{B}_{\Gamma_{C}}$. If there is ever a need to add shortest elements of a conjugacy class at some point in the algorithm, then we have a contradiction since there are no shortest elements in $\Gamma_{C}-\tilde{T}_{C}$.

Consider the first additions made to $\tilde{T}_{C}$ via $\mathfrak{B}_{\Gamma_{C}}$. Since for any $s \in S$, $\tilde{T}_{C}$ may be written as a linear combination of sums of $s$-conjugate elements for $s \in S$,
the only $s$-class elements which could possibly be incomplete are those of type $b_{d s}^{I I}=\tilde{T}_{d s}+\tilde{T}_{s d}+\xi_{s} \tilde{T}_{s d s}$ for some $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}$. Thus the only additions to $\tilde{T}_{C}$ will be of form $\xi_{s} \tilde{T}_{s d s}$ for some $s \in S$ and $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}$, and these are uniquely determined without reference to $\Gamma_{C}$.

Suppose by induction that after $k$ repeats of $\mathfrak{B}_{\Gamma_{C}}$, no shorter additions have been made. Equivalently, no choice has been made at any point in the construction so far: no reference to $\Gamma_{C}$ has been required.

Then all terms of length shorter than $m_{s}$ in $\mathfrak{B}_{\Gamma_{C}}^{k}\left(\tilde{T}_{C}\right)$ are in complete $s$-class elements for all $s \in S$ (as pointed out in Lemma (3.3.2)). Now suppose a shorter addition were required to complete the $s$-class element containing $\xi_{s} \mathfrak{m}_{i} \tilde{T}_{w}$ of length $l(w)=m_{s}$ in $\mathfrak{B}_{\Gamma_{C}}^{k}\left(\tilde{T}_{C}\right)$ for some $s \in S$. Then clearly $w=s d s$ for some $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}$, and we will need to add either $\xi_{s} \mathfrak{m}_{i} \tilde{T}_{d}$ or $\mathfrak{m}_{i}\left(\tilde{T}_{d s}+\tilde{T}_{s d}\right)$. The added element, being strictly shorter, will also reduce $m_{s}$ for $\mathfrak{B}_{\Gamma_{C}}^{k+1}\left(\tilde{T}_{C}\right)$, since there were no elements of that length or shorter incomplete in $\mathfrak{B}_{\Gamma_{C}}^{k}\left(\tilde{T}_{C}\right)$. (Note that if $\xi_{s}$ is not a factor of the coefficient $r$ of $\tilde{T}_{s d s}$ in $\mathfrak{B}_{\Gamma_{C}}^{k}\left(\tilde{T}_{C}\right)$, then it must be in the $s$-class element $r \tilde{T}_{d}+r \tilde{T}_{s d s}$, so we would need to add $\left.r \tilde{T}_{d}\right)$.

We will then need to add all of $\xi_{s} \mathfrak{m}_{i} \tilde{T}_{C_{d}^{d}}$ (resp. $\left.\mathfrak{m}_{i} \tilde{T}_{C_{d s}^{d s}}\right)$ if $\xi_{s} \mathfrak{m}_{i} \tilde{T}_{d}$ (resp. $\mathfrak{m}_{i}\left(\tilde{T}_{d s}+\right.$ $\left.\tilde{T}_{d s}\right)$ ) is added, by (3.3.1), without increasing $m_{s}$. If $d \notin C_{d, \min }$ (resp. $d s \notin$ $C_{d s, \min }$ ), then by (2.2.3) there is an element $u$ of $C_{d}^{d}$ (resp. $C_{d s}^{d s}$ ) which is of form $u=t d t$ for some $t \in S$ and $d$ distinguished. Thus if $d$ (resp. $d s$ ) is not minimal in its conjugacy class, we will require further strictly shorter additions via the algorithm. If $d$ or $d s$ is minimal, we have a contradiction since $\Gamma_{C}-\mathfrak{B}_{\Gamma_{C}}^{k}\left(\tilde{T}_{C}\right)$ contains no shortest elements with non-zero coefficient.

These shorter additions will continue, so long as $d \notin C_{\min }$ (resp. $d s \notin C_{\min }$ ). Thus, in a finite number of steps (since all of $C^{d}$ (resp. $C^{d s}$ ) will be added in a finite number of steps), we will add shortest elements of some conjugacy class, which is a contradiction.

Thus there is never a need to add strictly shorter elements at any point in the algorithm, and hence we never need to refer to the upper bound $\Gamma_{C}$.

Thus we have that under the condition that we start with $\tilde{T}_{C}$, the algorithm is well defined without reference to any upper bound at all. This motivates us to
make the following definition of a simpler "absolute" algorithm.
(3.3.4) Definition. Let $h \in \mathcal{H}^{+}$. Define the algorithm $\mathfrak{A}$ to conduct the following procedures.
(i) split $h$ into $h=h_{s}+h_{s}^{\prime}$ for each $s \in S$ such that $h_{s}$ is maximal in $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$ less than or equal to $h$;
(ii) if $h_{s}^{\prime}=0$ for all $s \in S$, stop;
$(i i)^{\prime}$ otherwise evaluate $m_{s}$ for each $s$ such that $h_{s}^{\prime} \neq 0$, and choose $s \in S$ such that $m_{s}$ is minimal;
(iii) add terms to $h$ which complete the $s$-class elements of those terms in $h_{s}^{\prime}$ of length $m_{s}$;
(iv) declare the new element to be $\mathfrak{A}(h)$, and repeat from (i) with the new element.
(3.3.5) Theorem. Let $C$ be a conjugacy class in $W$. Then the algorithm $\mathfrak{A}^{i}\left(\tilde{T}_{C}\right)$ is well defined for all $i \in \mathbb{N}$;

Proof. For $\mathfrak{A}$ to be well-defined we need to show that it does not add shorter at any point. But the algorithms $\mathfrak{A}$ and $\mathfrak{B}$ are identical if $\mathfrak{B}$ does not refer to an upper bound, and we have shown (in (3.3.3)) that this is the case when it is started on $\tilde{T}_{C}$. Thus $\mathfrak{A}$ does not add shorter at any point when started on $\tilde{T}_{C}$.
(3.3.6) Remark. It is sufficient to start $\mathfrak{A}$ on the sum of elements of $C_{\text {min }}$, since by (1.1.2) every element of $C$ can be obtained from a shortest element by a nondecreasing series of conjugations by simple reflections. Examples of the algorithm for types $B_{2}, A_{3}, B_{3}$ and $A_{4}$ are given in Chapter 5.

The point of the algorithm is that it provides us with a means of calculating these elements $\Gamma_{C}$, as shown in the following corollary.
(3.3.7) Corollary. $\mathfrak{A}^{n}\left(\tilde{T}_{C_{\text {min }}}\right)=\Gamma_{C}$ for some integer $n$.

Proof. From (3.3.5) and the remark above, there exists an integer $n$ for which $\mathfrak{A}^{n}\left(\tilde{T}_{C_{\text {min }}}\right)$ is well defined and in the centre, and less than $\Gamma_{C}$. By the minimality of $\Gamma_{C}$, we must have $\mathfrak{A}^{n}\left(\tilde{T}_{C_{\text {min }}}\right)=\Gamma_{C}$.

Remark. In order to use characters to calculate a primitive minimal positive central element, as done in other approaches to the question, one must calculate the coefficient of $\tilde{T}_{w}$ for every $w$ in $W$. The algorithm $\mathfrak{A}$ has the computational advantage that one only calculates coefficients for those terms whose coefficient is non-zero.

### 3.4 The minimal basis

(3.4.1) Lemma. If $w \in C_{\min }$ and $r \tilde{T}_{w} \leq h \in Z(\mathcal{H})^{+}$, then $r \Gamma_{C} \leq h$.

Proof. By (3.2.3) we have $r \tilde{T}_{C_{\text {min }}} \leq h$. The algorithm $\mathfrak{A}$ adds only same-length or longer when started on $\tilde{T}_{C_{\text {min }}}$, by (3.3.5), and by (3.1.2) all such additions are implications inside a central element. So, the same series of additions we would make if building $r \Gamma_{C}$ from $r \tilde{T}_{C_{\text {min }}}$ becomes a series of implications, giving us $r \Gamma_{C} \leq h$.
(3.4.2) Proposition. If $W$ is a Weyl group, then the set of class elements is the set of primitive minimal positive central elements of $\mathcal{H}$. That is,

$$
Z(\mathcal{H})_{\min }^{+}=\left\{\Gamma_{C} \mid C \in \operatorname{ccl}(W)\right\} .
$$

Proof. We will simply show that for any $h \in Z(\mathcal{H})^{+}$which is non-zero, there exists a class element $\Gamma_{C}$ such that $r \Gamma_{C} \leq h$ for some $r \in R^{+}$.

Since $h$ is non-zero, by (3.1.3) there is a shortest element $w$ of some conjugacy class $C$ such that $\tilde{T}_{w}$ has non-zero coefficient $r \in R^{+}$in $h$. Then by (3.4.1) we must have $r \Gamma_{C} \leq h$. This completes the proof.
(3.4.3) Theorem. $Z(\mathcal{H})_{\min }^{+}$is an $R$-basis for $Z(\mathcal{H})$.

Proof. The linear independence of the elements in $Z(\mathcal{H})_{\min }^{+}$can be seen by specializing to $\xi_{s}=0$ for all $s \in S$ using the same inductive argument as in (1.2.2).

It remains to show spanning. We begin by showing that $Z(\mathcal{H})^{+}$is spanned by the set $Z(\mathcal{H})_{\text {min }}^{+}$over $R^{+}$.

Let $h \in Z(\mathcal{H})^{+}$. If $h \in Z(\mathcal{H})_{\min }^{+}$, then we are done, so suppose otherwise. Then $h$ is either not minimal, or is minimal with a factor of $\mathfrak{m}_{i}$ for some monomial $\mathfrak{m}_{i}$.

If the latter, then we are done, as $h$ is an $R^{+}$-multiple of an element of $Z(\mathcal{H})_{\min }^{+}$. If the former, then there exists a minimal element $a_{1} \in\left(Z(\mathcal{H})^{+}, \leq\right)$such that $0 \neq a_{1}<h$. Let $k \in \mathbb{N}$ be maximal such that $\mathfrak{m}_{k}$ is a factor of $a_{1}$. Then we can write $a_{1}=\mathfrak{m}_{k} a_{1}^{\prime}$, with $a_{1}^{\prime} \in Z(\mathcal{H})_{\text {min }}^{+}$.

Then $h=h_{1}+\mathfrak{m}_{k} a_{1}^{\prime}$, where $h_{1} \in Z(\mathcal{H})^{+}$, and $h_{1}<h$.
We can now repeat the process for $h_{1}$, removing an $R^{+}$multiple of an element of $Z(\mathcal{H})_{\min }^{+}$and staying in $Z(\mathcal{H})^{+}$. Thus we may continually reduce $h$ by nontrivial multiples of elements of $Z(\mathcal{H})_{\min }^{+}$. This sequence of reductions will finish in a finite number of steps, as $h \in \sum_{w \in W}\left[\bigoplus_{j \in \mathbb{N}, \mathfrak{m}_{j} \in \operatorname{Mon}_{j}} \mathbb{N m}_{j}\right] \tilde{T}_{w}$, giving us $h$ as an $R^{+}$-linear combination of elements of $Z(\mathcal{H})_{\text {min }}^{+}$.

Now suppose $h \in Z(\mathcal{H})$. Then $h=h^{+}+h^{-}$, where $h^{+} \in \mathcal{H}^{+}$, and $-h^{-} \in \mathcal{H}^{+}$. Choose $w \in W$ such that $l(w)$ is minimal for the terms in $h^{-}$. Then $w \in C_{\min }$ for some conjugacy class $C$ of $W$. If $\tilde{T}_{w}$ has coefficient $-r$ in $h^{-}$for some $r \in R^{+}$, then so does $\tilde{T}_{C_{w, \text { min }}}$, by (3.2.3), because they have the same coefficient in $h$. If we then add $r \Gamma_{C}$ to $h$, then $-r \tilde{T}_{w}$ is no longer a term of $h+r \Gamma_{C}$. We may proceed in this way to remove all negative terms in $h$, by adding an $R^{+}$-linear combination of the $\Gamma_{C}$, giving us

$$
h+\sum_{C} r_{C} \Gamma_{C} \in Z(\mathcal{H})^{+}
$$

for some coefficients $r_{C}$ in $R^{+}$.
Then by our above work, we may write $h+\sum_{C} r_{C} \Gamma_{C}$ as an $R^{+}$-linear combination of the $\Gamma_{C}$ also, so that $h+\sum_{C} r_{C} \Gamma_{C}=\sum_{C} r_{C}^{\prime} \Gamma_{C}$, giving us

$$
h=\sum_{C}\left(r_{C}^{\prime}-r_{C}\right) \Gamma_{C},
$$

so that $h$ is in the $R$-span of the class elements.
We can now collate these results into a proof of our main theorem for centres, in the case that $W$ is a Weyl group:

Proof of (2.2.1). (i) is (3.4.3), and (ii) is (3.4.2) and (3.2.6).

### 3.5 Remarks

All results of the past chapter have been stated for centres when $W$ is a Weyl
group, as this is the most important case, and in the case of conjugacy classes we have the result of Geck and Pfeiffer (1.1.2) to help us.

However, as commented before, many of the results are valid for $J$-conjugacy classes, with $J \subseteq S$. Specifically, all results of this chapter are valid for $J$-conjugacy classes and centralizers $Z_{\mathcal{H}}\left(\mathcal{H}_{J}\right)$ if a result analogous to (1.1.2) can be proved. Similarly, if $W$ is non-crystallographic, we need an result analogous to (1.1.2) to deal with this case.

For example, if the existence of an element $\Gamma_{\mathfrak{C}} \in Z_{\mathcal{H}}\left(\mathcal{H}_{J}\right)$ could be shown, with the properties that it specialized to $\tilde{T}_{\mathcal{C}}$, and contained no other shortest elements, then the algorithms of 3.3 can be defined relative to such an element using $s$-class elements for $s \in J$ and the rest of that section follows. The only external result needed in section 3.2 is (1.1.2).

In the next chapter we prove analogies to (1.1.2) for certain cases of $J$-conjugacy classes in types $A$ and $B$, in chapter 5 for some non-crystallographic types, and for some cases of small $J$, and in 6.4 we make a conjecture on a complete generalization.

### 3.6 Reverting to a basis over $\mathbb{Z}\left[q, q^{-1}\right]$

We now demonstrate how to obtain the analogous basis for the centre over $\mathbb{Z}\left[q, q^{-1}\right]$. Of course one can immediately obtain corresponding central elements over $\mathbb{Z}\left[q^{1 / 2}, q^{-1 / 2}\right]$ by setting $q=q_{s}$ for all $s \in S$, and substituting $\xi_{s}=q^{1 / 2}-q^{-1 / 2}$ and $\tilde{T}_{w}=q^{-\frac{l(w)}{2}} T_{w}$ in $\Gamma_{C}$ for each conjugacy class $C$ and $s \in S$. We will show that these are either over $\mathbb{Z}\left[q, q^{-1}\right]$ or are $q^{-1 / 2}$-multiples of elements over $\mathbb{Z}\left[q, q^{-1}\right]$. We begin with a result on the coefficients of terms in $\Gamma_{C}$ :
(3.6.1) Lemma. Suppose $\mathfrak{m}_{i} \tilde{T}_{w} \leq \Gamma_{C}$ for some monomial $\mathfrak{m}_{i}$ of order $i$ in the $\xi_{s}$ and some conjugacy class $C$. Then
(i) $i$ is even if and only if $l(w)=l_{C}+2 k$ for some integer $k \geq 0$, and
(ii) $i$ is odd if and only if $l(w)=l_{C}+2 k+1$ for some $k \geq 0$.

Furthermore, we always have $i \leq l(w)-l_{C}$.
Proof. Consider how additions of terms of different lengths and different coefficients may arise due to the algorithm. The only way the order of the coefficient
is increased is by adding $\xi_{s} r \tilde{T}_{s d s}$ to complete the $s$-class element $r b_{d s}^{I I}$ (for $r \in R$ ), and this addition is also the only way the length of the element can be increased by an odd number - one. All other completions (for $s$-class elements $b_{d}^{I I}$, or of form $\tilde{T}_{d s}$ in $b_{d s}^{I I}$ ) maintain the power of $\xi_{s}$ and add terms of length two greater than that already present.

Thus, even (resp. odd) orders of the monomial coefficient and terms of length an even (resp. odd) difference from $l_{C}$ arise only by an even (resp. odd) sequence of $b_{d s}^{I I}$ completions of form $\xi_{s} \tilde{T}_{s d s}$ (interspersed perhaps with $b_{d}^{I I}$ completions, which do not alter the coefficient).

An increase by one in the order of the monomial coefficient are linked to an increase by one in the length of the word. Thus the maximum order of the monomial possible in the coefficient of $\tilde{T}_{w}$ would be if we were to increase the order by one for every increase by one in word length, from the shortest word in $C$ up to the addition of $\mathfrak{m}_{i} \tilde{T}_{w}$. This gives a maximum increase in the order of the coefficient (from order zero - a coefficient of one) of $l(w)-l_{C}$.

If we write $\Gamma_{C, q}$ for the image of $\Gamma_{C}$ in the injection $\mathcal{H} \rightarrow \mathcal{H}_{q} \rightarrow \mathcal{H}_{\mathbb{Z}\left[q^{1 / 2}, q^{-1 / 2}\right]}$ (defined by setting $\xi_{s}=q^{1 / 2}-q^{-1 / 2}$ and $\tilde{T}_{w}=q^{-\frac{l(w)}{2}} T_{w}$ ), then we have the following consequence of the lemma:
(3.6.2) Proposition. If $l_{C}$ is even, then $\Gamma_{C, q} \in \mathcal{H}_{\mathbb{Z}\left[q, q^{-1}\right]}$. If $l_{C}$ is odd, then $q^{-1 / 2} \Gamma_{C, q} \in \mathcal{H}_{\mathbb{Z}\left[q, q^{-1}\right]}$.
Proof. Every term in $\Gamma_{C}$ is of form $\mathfrak{m}_{i} \tilde{T}_{w}$ which gives $\left(q^{1 / 2}-q^{-1 / 2}\right)^{i} q^{-l(w) / 2} T_{w}$ in $\Gamma_{C, q}$.

If $l_{C}$ is even, then by (3.6.1) we will have either $i$ even and $l(w)$ even, or $i$ odd and $l(w)$ odd. In either case we have $i+l(w)$ even, and so

$$
\begin{aligned}
\left(q^{1 / 2}-q^{-1 / 2}\right)^{i} q^{-l(w) / 2} & =q^{-i / 2}(q-1)^{i} q^{-l(w) / 2} \\
& =q^{-\frac{1}{2}(i+l(w))}(q-1)^{i} \in \mathbb{Z}\left[q, q^{-1}\right] .
\end{aligned}
$$

If $l_{C}$ is odd, again by (3.6.1) we have either $i$ even and $l(w)$ odd, or $i$ odd and $l(w)$ even. Then in either case we have $i+l(w)$ odd, and so

$$
q^{-1 / 2}\left(q^{1 / 2}-q^{-1 / 2}\right)^{i} q^{-l(w) / 2}=q^{-1 / 2(1+i+l(w))}(q-1)^{i} \in \mathbb{Z}\left[q, q^{-1}\right]
$$

which completes the proof.
(3.6.3) Corollary. The set $\left\{\Gamma_{C, q} \mid C \in \operatorname{ccl}(W)\right.$, and $l_{C}$ even $\} \cup\left\{q^{-1 / 2} \Gamma_{C, q} \mid\right.$ $C \in \operatorname{ccl}(W)$, and $l_{C}$ odd $\}$ is a $\mathbb{Z}\left[q, q^{-1}\right]$ basis for $Z\left(\mathcal{H}_{\mathbb{Z}\left[q, q^{-1}\right]}\right)$.

## Chapter 4 <br> Centralizers of principal parabolic subalgebras

We devote this chapter to proving an analogy of (2.2.1) in certain cases of centralizers $Z_{\mathcal{H}}\left(\mathcal{H}_{J}\right)$, for $J \subseteq S$. Specifically, we deal here with when $W$ is of type $A$ or $B$, and the subset $J$ of $S$ is "principal".

As pointed out in 3.5, the remaining step in this agenda is to prove an analogy of (1.1.2) for the $J$-conjugacy classes in the above cases. The challenge in proving such an analogy in general is that for a $J$-conjugacy class $\mathfrak{C}$ contained in the double coset $W_{J} d W_{J}$, we need to describe how $d \in \mathfrak{D}_{J, J}$ interacts with the elements of $J$. That is, for which $s_{i}$ and $s_{j}$ in $J$ we have $s_{i} d=d s_{j}$. The use of "chains" is a natural tool to help describe this, and in the principal case we are able to describe this interaction explicitly.

### 4.1. Chains in Weyl groups of types $A$ and $B$

We will use the following notation for the Weyl groups of types $A$ and $B$ :


We denote by $W_{i}$ the subgroup of $W$ generated by all simple reflections to the left of $s_{i}$ in the Dynkin diagram, including $s_{i}$, so that $W_{i}=W\left(A_{i}\right)$ or $W\left(B_{i+1}\right)$. We use this notation to avoid the need to differentiate the cases at $s_{0}$. We will also need the convention that $W_{0}=1$ in type $A_{n}$, and $\langle t\rangle$ in type $B_{n}$, and if $i<0$, then $W_{i}=1$ in all cases. Similarly, for convenience $s_{0}$ will denote 1 in type $A_{n}$ and $t$ in type $B_{n}$. So $W_{i}$ is a principal parabolic subgroup of $W$.

Define the following elements, for $0 \leq i<j \leq n$ :

$$
\begin{aligned}
r_{i, j} & =s_{i} s_{i+1} \ldots s_{j-1} s_{j} s_{j-1} \ldots s_{i+1} s_{i} \\
r_{j, i} & =s_{j} s_{j-1} \ldots s_{i+1} s_{i} s_{i+1} \ldots s_{j-1} s_{j}
\end{aligned}
$$

We will write $r_{i, t}$ for $r_{i, 0}=s_{i} s_{i-1} \ldots s_{1} t s_{1} \ldots s_{i-1} s_{i}$ in type $B_{n}$.
The definitions of primitive and elementary chains, sources, and targets given below are due to Brieskorn and Saito ([BS]), and have been translated by Clare

Coleman, Ruth Corran, John Crisp, David Easdown, Bob Howlett, David Jackson, and Arun Ram in [CCCEHJR].

Define an $s_{i}$-chain to be a reduced word representing an element $w \in W$ such that $s_{i} w=w s_{j}$ for some $s_{j} \in J$. We say $w$ has source $s_{i}$ and target $s_{j}$.

An $s_{i}$-chain $w$ of target $s_{i}$ is said to be primitive if every simple reflection in the reduced expressions of $w$ commutes with $s_{i}$. Note that the set of simple reflections forming each reduced expression for $w$ is the same, so this definition makes sense. An $s_{i}$-chain $w$ of target $s_{j}$ is said to be elementary if it contains no primitive $s_{i}$-chains, and may not be written as a product of shorter elementary chains.

Some preliminary facts from [BS] are:
(4.1.1). If $w \in W$ satisfies $s_{i} w=w s_{j}$ for some $s_{i}, s_{j} \in S$, then $w$ has a reduced expression as a product of primitive and elementary chains.
(4.1.2). The only elementary $s_{i}$-chains in a Coxeter group $W$ are products of form $s_{j} s_{i} s_{j} \ldots$ (where there are $<m_{s_{i} s_{j}}$ terms in the product) with $m_{s_{i} s_{j}} \geq 3$.

We will represent an $s_{i}$-chain $w$ of target $s_{j}$ by $s_{i} \xrightarrow{w} s_{j}$. For types $A$ and $B$ we have the following summary of elementary $s_{i}$-chains for $1 \leq i<n$ :

$$
\begin{aligned}
& s_{i} \xrightarrow{s_{i+1} s_{i}} s_{i+1}, \\
& s_{i+1} \xrightarrow{s_{i} s_{i+1}} s_{i}, \text { and } \\
& t \xrightarrow{s_{1} t s_{1}} t \text { and } s_{1} \xrightarrow{t s_{1} t} s_{1} \text { in type } B .
\end{aligned}
$$

Clearly most of these cases have different source and target (the exception being those in type $B$ around the generator $t$ ). We will introduce a convenient shorthand by saying an elementary chain goes "up" if its target is further to the right of the Dynkin diagram (as drawn above) than its source, and "down" if its target is to the left of its source.

Given that we may decompose any chain $w$ into a sequence of primitive and elementary chains, we may define a sequence consisting of the source of $w$ followed by the targets of the elementary chains in $w$. We will call this sequence the target sequence. Adjacent entries of the target sequence will vary by at most one generator, since the elementary chains go up or down by at most one generator. If $w$ is an $s_{i}$-chain of target $s_{j}$, then its inverse $w^{-1}:=w_{r e v}$ is an $s_{j}$-chain of target $s_{i}$.

For example, in type $B$, the reduced word $s_{2} s_{3} s_{1} s_{2} t s_{1} t s_{2} s_{1}$ is a chain of source $s_{3}$ and target $s_{2}$, and the corresponding target sequence is $\left[s_{3}, s_{2}, s_{1}, s_{1}, s_{2}, s_{3}\right]$. One property of the target sequence for an $s_{i}$-chain $w$ of target $s_{j}$ is that the sequence in reverse order is the target sequence for $w_{r e v}$.

An element $s_{i}$ of the target sequence is called a local minimum if the first distinct elements to either side of $s_{i}$ are both greater than $s_{i}$ (in the order induced by the Dynkin diagram). Similarly, a local maximum is an element of the target sequence greater than the first distinct elements to either side of it.
(4.1.3) Remark. In the case of primitive $s_{i}$-chains, note that the set of all primitive $s_{i}$-chains is $W_{i-2} \times\left\langle s_{i+2}, \ldots, s_{n}\right\rangle$.

We start with a couple of results for certain chains in Weyl groups of type $A$ and $B$. The following lemma shows that any local minimum which is not $s_{1}$ in type $B$ can be transformed using the Coxeter relations into a local maximum.
(4.1.4) Lemma. Suppose $s_{i} \neq s_{2}$ in type $B$. If $w \in W(B)$ has an expression as an $s_{i}$-chain going down to $s_{i-1}$ and back up to $s_{i}$ then it has an expression going first up to $s_{i+1}$ and then back down to $s_{i}$.

Proof. Let $w$ be such an $s_{i}$-chain. Then because $s_{i-1}$ is not $s_{1}$ in type $B$, there are no elementary $s_{i-1}$-chains of target $s_{i-1}$, and so $w$ has target sequence $\left[s_{i}, s_{i-1}, s_{i}\right]$. The lemma claims that if we assume that $s_{i-1}$ is not $s_{1}$ in type $B$, then we can rewrite $w$ in another reduced form having target sequence $\left[s_{i}, s_{i+1}, s_{i}\right]$.

Between each element of the target sequence there is exactly one elementary chain, so we may write $w$ as the product of chains

$$
w=C_{1} C_{2} C_{3} C_{4} C_{5}
$$

where $C_{2}$ is an elementary $s_{i}$-chain of target $s_{i-1}, C_{4}$ is an elementary $s_{i-1}$-chain of target $s_{i}, C_{1}$ and $C_{5}$ are primitive $s_{i}$-chains, and $C_{3}$ is a primitive $s_{i-1}$-chain. We immediately have $C_{2}=s_{i-1} s_{i}$, and $C_{4}=s_{i} s_{i-1}$, and for the primitive chains, we have $C_{1}, C_{5} \in W_{i-2} \times\left\langle s_{i+2}, \ldots, s_{n}\right\rangle$. Then all primitive $s_{i-1}$-chains are in $W_{i-3} \times\left\langle s_{i+1}, \ldots, s_{n}\right\rangle$, where we let $W_{i-3}=1$ if $i-3<0$ in all cases. But since $W_{i-3}$ commutes with $C_{2}$ and $C_{4}$, we may write $w$ in the following form:

$$
w=w_{1} s_{i-1} s_{i} w_{2} s_{i} s_{i-1} w_{3},
$$

where $w_{1}, w_{3} \in W_{i-2} \times\left\langle s_{i+2}, \ldots, s_{n}\right\rangle$ and $w_{2} \in\left\langle s_{i+1}, \ldots, s_{n}\right\rangle$.
Now the subgroup $\left\langle s_{i+1}, \ldots, s_{n}\right\rangle$ is isomorphic to the Coxeter group of type $A_{n-i}$, under the correspondence $s_{k} \leftrightarrow s_{k-i}$, so we may write any element of $\left\langle s_{i+1}, \ldots, s_{n}\right\rangle$ in reduced form with a single occurence of $s_{i+1}$. This gives us $w_{2}=w_{2}^{\prime} s_{i+1} w_{2}^{\prime \prime}$ where $w_{2}^{\prime}, w_{2}^{\prime \prime} \in\left\langle s_{i+2}, \ldots, s_{n}\right\rangle$, and both these commute with the elementary chains $s_{i-1} s_{i}$ and $s_{i} s_{i-1}$. So in fact we may write

$$
w=w_{1} w_{2}^{\prime} s_{i-1} s_{i} s_{i+1} s_{i} s_{i-1} w_{2}^{\prime \prime} w_{3} .
$$

Let $C_{1}^{\prime}=w_{1} w_{2}^{\prime}$, and $C_{5}^{\prime}=w_{2}^{\prime \prime} w_{3}$. Both these are primitive $s_{i}$-chains. Then use of the braid relation gives

$$
w=C_{1}^{\prime} s_{i+1} s_{i} s_{i-1} s_{i} s_{i+1} C_{5}^{\prime}
$$

which has target sequence $\left[s_{i}, s_{i+1}, s_{i}\right]$, which proves the lemma.
(4.1.5) Proposition. Let $w$ be a chain going both up and down. Then
(i) in type A, w may be written with a unique local maximum or a unique local minimum;
(ii) in type $B$, $w$ may be written such that the only local minima are $s_{1}$, and between each such minimum is a unique local maximum.

Proof. In type $A$, the above lemma shows that every local minimum may be transformed into a local maximum, and repeated applications of this lemma will eliminate any local minima. It is easy to construct an algorithm to do this, for instance by removing first the right-most local minimum, as done pictorially below:


Similarly, every local maximum may be transformed into a local minimum, because in type $A$ any subgroup $W_{i}$ is also of type $A_{i}$.

For type $B$, simply apply the previous lemma to eliminate all local minima between minima of $s_{1}$, as with type $A$.

The following lemma is trivial.
(4.1.6) Lemma. Let $w$ be an $s_{i}$-chain of target $s_{i+k}$, for $k \geq 2$, which only goes up. Then
(i) we may write:

$$
w=C_{1} w^{\prime} C_{2}
$$

with $w^{\prime}=\left(s_{i+1} s_{i}\right)\left(s_{i+2} s_{i+1}\right) \ldots\left(s_{i+k} s_{i+k-1}\right), C_{1} \in\left\langle s_{i+2}, \ldots, s_{n}\right\rangle$ a primitive $s_{i}-$ chain, and $C_{2} \in\left\langle s_{1}, \ldots, s_{i+k-2}\right\rangle$ a primitive $s_{k}$-chain, and
(ii) $w_{\text {rev }}^{\prime}$ is an $s_{j}$-chain of target $s_{j+2}$, for all $i \leq j \leq i+k-2$.

We will need the following normal form for the longest word in Weyl groups of type $B$. Note that the longest element $u$ of any Coxeter group is the unique element characterized by the property that $l(s u)=l(u)-1$ for all $s \in S$ (see for example [H;(1.8)])
(4.1.7) Lemma. The longest word in $W\left(B_{n+1}\right)$ is

$$
\begin{aligned}
w_{n} & =r_{n, t} w_{n-1} \\
& =r_{n, t} r_{n-1, t} \ldots s_{1} t s_{1} t .
\end{aligned}
$$

Proof. The lemma is true for both $B_{1}$ and $B_{2}$, where the longest words are respectively $t$ and $s_{1} t s_{1} t$, so we assume inductively that it is true for $n-1$. We need to show that $r_{n, t} w_{n-1}$ is longest in $W\left(B_{n+1}\right)$. Note that $r_{n, t} \in Z_{W\left(B_{n+1}\right)}\left(W\left(B_{n}\right)\right)$, so $r_{n, t} w_{n-1}=w_{n-1} r_{n, t}$. Since $w_{n-1}$ is longest in $W\left(B_{n}\right)$, we have that $l\left(s w_{n-1}\right)=$ $l\left(w_{n-1}\right)-1$ for all $s \in\left\{t, s_{1}, \ldots, s_{n-1}\right\}$. Then we must also have $l\left(s w_{n}\right)=l\left(w_{n}\right)-1$ for all $s \in S \backslash\left\{s_{n}\right\}$, by the previous two points. However $l\left(s_{n} w_{n}\right)=l\left(w_{n}\right)-1$ also, since $l\left(s_{n} r_{n, t}\right)=l\left(r_{n, t} s_{n}\right)=l\left(r_{n, t}\right)-1$ and $w_{n-1}$ commutes with $r_{n, t}$. Thus $w_{n}$ is the longest word in $W\left(B_{n+1}\right)$.

### 4.2. Double cosets of principal parabolic subgroups

We now obtain some results specifically for those chains which are double coset representatives of principal parabolic subgroups (or double coset representatives which are also chains).
(4.2.1) Lemma. Let $J$ be principal, and let $d \in \mathfrak{D}_{J, J}$ be a chain. Then $d$ must start up and end down.

Proof. Let $d \in \mathfrak{D}_{J, J}$ be an $s_{i}$-chain which starts down. Then we may write $d=$ $C_{1} C_{2} \ldots$, where $C_{1}$ is a primitive $s_{i}$-chain, and $C_{2}$ is an elementary $s_{i}$-chain of target $s_{i-1}$. By (4.1.3) we must have $C_{1} \in W_{i-2} \times\left\langle s_{i+2}, \ldots, s_{n}\right\rangle$, and $C_{2}=s_{i-1} s_{i}$. But since $J$ is principal, $W_{i-2} \subset W_{J}$, so we must have $C_{1} \in\left\langle s_{i+2}, \ldots, s_{n}\right\rangle$ if $d$ is to be distinguished. But then $d=C_{1} s_{i-1} s_{i} \cdots=s_{i-1} s_{i} C_{1} \ldots$, and so $d$ cancels on the left with $s_{i-1} \in J$, a contradiction. Similar working gives that the chain must end down.
(4.2.2) Proposition. Let $J$ be principal, and let $d \in \mathfrak{D}_{J, J}$ be an $s_{i}$-chain, for $1 \leq i \leq n$. Then
(i) In type $A$, $d$ is a primitive $s_{i}$-chain;
(ii) In type B, if d goes up and then down, it has a unique local maximum $s_{i+1}$, and has target $s_{i}$.

Proof. (i) From (4.1.7), in type $A$ we may write any non-primitive chain starting down, and so by (4.2.1) the chain cannot be a distinguished double coset representative.
(ii) By (4.2.1), $d$ must start up, and so it has a first point at which it goes down - a first maximum.

The argument is the same for any maximum $s_{m}$ for $m$ greater than or equal to $i+2$, so suppose that the maximum is $s_{i+2}$. Using (4.1.6), we may write

$$
d=C_{1}\left(s_{i+1} s_{i}\right)\left(s_{i+2} s_{i+1}\right) C_{2}\left(s_{i+1} s_{i+2}\right)\left(s_{i} s_{i+1}\right) C_{3}
$$

for some $k \leq i+1$, with $C_{1} \in\left\langle s_{i+2}, \ldots, s_{n}\right\rangle$ a primitive $s_{i}$-chain, $C_{2} \in W_{i}$ a primitive $s_{i+2}$-chain, and $C_{3} \in\left\langle s_{k+2}, \ldots s_{n}\right\rangle$ a primitive $s_{k}$-chain. We will look at the structure of $C_{2}$.

Since $C_{2}$ is in $W\left(B_{i+1}\right)$, we may think of it as a subword of the longest word $w_{i}=r_{i, t} w_{i-1}$. Thus, to begin with, we can be sure that it is possible to write $C_{2}$ with at most two occurences of $s_{i}$. There must also be at least one such occurence, since otherwise $d$ would not be reduced (the $s_{i+1}$ on either side of $C_{2}$ would commute with $C_{2}$ and cancel). If it is not possible to write $C_{2}$ with only one occurence of $s_{i}$, then since $C_{2}$ is a subword of $w_{i}$, it must cancel on the left with $s_{i}$. But $s_{i}$ may be commuted through to the left of $d$, and we may thus assume there is only one occurence of $s_{i}$ in $C_{2}$. Using the same logic, we may assume that $C_{2}$ can be written as a subword of $\left(s_{i-1} \ldots s_{1} t s_{1} \ldots s_{i}\right) w_{i-1}$.

Further, we must have $s_{i-1} s_{i} \leq C_{2}$, since otherwise the $s_{i}$ would be able to commute to the left, leaving the $s_{i+1}$ on each side of $C_{2}$ to cancel with each other. Since $d$ is distinguished and $J$ is principal, we must then have either $C_{2}=$ $\left(s_{i-1} s_{i-2} \ldots s_{1} t s_{1} \ldots s_{i-2} s_{i-1}\right) u$ or $\left(s_{i-1} s_{i}\right) u$, where $u$ is a subword of $w_{i-1}$.

We now have that $d$ may be written either in the form

$$
\begin{aligned}
& d=C_{1}\left(s_{i+1} s_{i}\right)\left(s_{i+2} s_{i+1}\right)\left(s_{i-1} \ldots s_{1} t s_{1} \ldots s_{i}\right) u\left(s_{i+1} s_{i+2}\right) \ldots\left(s_{k} s_{k+1}\right) C_{3}, \text { or } \\
& d=C_{1}\left(s_{i+1} s_{i}\right)\left(s_{i+2} s_{i+1}\right)\left(s_{i-1} s_{i}\right) u\left(s_{i+1} s_{i+2}\right) \ldots\left(s_{k} s_{k+1}\right) C_{3}
\end{aligned}
$$

for $u$ a subword of $w_{i-1}$. In either case, we have $s_{i+1}$ commutes with $u$, and in fact $s_{i+1}$ passes through to the left in both cases, emerging as $s_{i-1}$, which contradicts that $d$ is distinguished. Thus $d$ goes up at most one at its leftmost end.

We now suppose that after going up exactly one, $d$ goes down at least two steps. Then we may write $d=C_{1}\left(s_{i+1} s_{i}\right) C_{2}\left(s_{i} s_{i+1}\right)\left(s_{i-1} s_{i}\right) \ldots\left(s_{k} s_{k+1}\right) \ldots$ where $k<i$,
and $C_{2} \in W_{i-1}$. Now we have slightly more control over $C_{2}$, as any element of $W_{i-2}$ will commute to the left and cancel, contradicting that $d$ is distinguished. Thus, looking at $C_{2}$ as a subword of $w_{i-1}=r_{i-1, t} w_{i-2}$, we have either $C_{2}=r_{i-1, t} u$ or $s_{i-1} u$, for $u$ a subword of $w_{i-2}$. If the latter case, and $C_{2}=s_{i-1} u$, then we would have

$$
\begin{aligned}
d & =C_{1}\left(s_{i+1} s_{i}\right) s_{i-1} u\left(s_{i} s_{i+1}\right)\left(s_{i-1} s_{i}\right) \ldots \\
& =C_{1} s_{i-1} s_{i} s_{i+1} s_{i} s_{i-1} u\left(s_{i-1} s_{i}\right) \ldots
\end{aligned}
$$

which cancels on the left with $s_{i-1}$ - a contradiction. Thus $C_{2}=r_{i-1, t} u$ in type $B_{n+1}$. But $r_{i-1, t}$ is in the centralizer of $W\left(B_{i-1}\right)$ in $W\left(B_{i}\right)$. In particular, if $u \neq 1$ then $d$ will cancel on the left by an element of $J$. Thus $u=1$, and $C_{2}=r_{i-1, t}$. Thus (in type $B_{n+1}$ ),

$$
\begin{aligned}
d & =C_{1}\left(s_{i+1} s_{i}\right) r_{i-1, t}\left(s_{i} s_{i+1}\right)\left(s_{i-1} s_{i}\right) \ldots \\
& =C_{1} r_{i+1, t}\left(s_{i-1} s_{i}\right) \ldots \\
& =s_{i-1} s_{i} C_{1} r_{i+1, t} \ldots
\end{aligned}
$$

which is a contradiction.
(4.2.3) Corollary. Let $J$ be principal.
(i) If $d \in \mathfrak{D}_{J, J}$ is in type $B$ and is a non-primitive $s_{i}$-chain for $1 \leq i \leq n$, we may write

$$
d=C_{1} r_{i+1, t} C_{2}
$$

where $C_{1}, C_{2} \in\left\langle s_{i+2}, \ldots, s_{n}\right\rangle$.
(ii) Suppose $d \in \mathfrak{D}_{J, J}$ is of type $A$ or $B$, and $W_{J}^{d} \cap W_{J}=W_{J^{\prime}}$. If $s \in J^{\prime}$ then $d s=s d$.

Proof. (i) follows from the proof of (4.2.2). For (ii), it only remains to show for $s=t$ in type $B$. But in this case, the only elementary $t$-chain is $s_{1} t s_{1}$, and this has target $t$. Thus any $t$-chain will have target $t$.
(4.2.4) Theorem. Suppose $W$ is of type $A$ or $B, J \subseteq S$ is principal, $d \in \mathfrak{D}_{J, J}$ and $W_{J}^{d} \cap W_{J}=W_{J^{\prime}}$ for some $J^{\prime} \subseteq J$. Then $J^{\prime}$ is principal in $S$.

Proof. If $s_{i} \in J^{\prime}$, it means that $d$ is an $s_{i}$-chain of target in $J$. To show the corollary, it is sufficient to show that $s_{i} \in J^{\prime}$ implies $s_{i-1} \in J^{\prime}$. In type $A$, this
is clear, since if $d$ is an $s_{i}$-chain then it is primitive, and so $d \in\left\langle s_{i+2}, \ldots, s_{n}\right\rangle$, by (4.2.2)(i).

From (4.2.3), in type $B$, we have $d=C_{1} r_{i+1, t} C_{2}$, where $C_{j} \in\left\langle s_{i+2}, \ldots, s_{n}\right\rangle$. The result follows.

### 4.3. Reducibility in $J$-conjugacy classes

We will show in this section that if $J$ is a principal subset of $S$ and $W$ is of type $A$ or $B$, then all $J$-conjugacy classes of $W$ are reducible. This provides a generalization of (1.1.2) and consequently of a generalization of (2.2.1).

Now, we have shown that if $W_{J}^{d} \cap W_{J}=W_{J^{\prime}}$ and $s_{i} \in W_{J^{\prime}}$ for $i \geq 1$, then $s_{i}$ commutes with $d$, and so do all $s_{j}$ for all $j \leq i$.
(4.3.1) Lemma. Let $W$ be of type $A$ or $B$, let $J \subseteq S$ be principal, let $d \in \mathfrak{D}_{J, J}$, and suppose $W_{J}^{d} \cap W_{J}=W_{J^{\prime}}$. Then dw is shortest in its J-conjugacy class if and only if $w \in W_{J}$ is shortest in its $J^{\prime}$-conjugacy class.

Proof. Suppose $d w$ is shortest in its $J$-conjugacy class, and $w$ is not shortest in its $J^{\prime}$-conjugacy class. Then there is an element $u$ of $W_{J^{\prime}}$ with the property that $l\left(u w u^{-1}\right)<l(w)$. But then $u d w u^{-1}=d u w u^{-1}$, since $d \in Z_{W}\left(W_{J^{\prime}}\right)$, and so $l\left(u d w u^{-1}\right)=l\left(d u w u^{-1}\right)=l(d)+l\left(u w u^{-1}\right)<l(d)+l(w)=l(d w)$ which contradicts that $d w$ is shortest in its $J$-conjugacy class.

On the other hand, suppose $w$ is shortest in its $J^{\prime}$-conjugacy class, and let $u \in W_{J}$ be arbitrary. We may write $u=u_{1} u_{2}$ with $l(u)=l\left(u_{1}\right)+l\left(u_{2}\right)$ where $u_{2} \in W_{J^{\prime}}$ and $u_{1} \in \mathfrak{D}_{W_{J} / W_{J^{\prime}}}$. Then $u d w u^{-1}=u_{1} d u_{2} w u_{2}^{-1} u_{1}^{-1}$, and so

$$
\begin{aligned}
l\left(u d w u^{-1}\right) & =l\left(u_{1} d u_{2} w u_{2}^{-1} u_{1}^{-1}\right) \\
& =l\left(u_{1}\right)+l(d)+l\left(u_{2} w u_{2}^{-1} u_{1}^{-1}\right) \\
& \geq l\left(u_{1}\right)+l(d)+l\left(u_{2} w u_{2}^{-1}\right)-l\left(u_{1}^{-1}\right) \\
& =l(d)+l\left(u_{2} w u_{2}^{-1}\right) \\
& \geq l(d)+l(w) \\
& =l(d w)
\end{aligned}
$$

since $d$ is distinguished, and $w$ is minimal in its $J^{\prime}$-conjugacy class. Thus $d w$ is minimal in its $J$-conjugacy class.

Now we give a generalization of (1.1.2)(i).
(4.3.2) Theorem. Let $W$ be of type $A$ or $B$, and let $J$ be principal in $S$. Then every $J$-conjugacy class in $W$ is reducible.

Proof. We proceed by induction on the size of $W$.
Firstly we establish that the result holds for $W_{1}=W\left(A_{1}\right)$ and $W\left(B_{2}\right)$. In these cases the solution for the centres (1.1.2) and the case $|J|=1$ described in 3.1 provides the solution.

Suppose then that it holds for all $W_{i}$ with $i \leq k$. We need to show for $W_{k+1}$. We may suppose that $J \subset S$, since the case where $J=S$ is the case for conjugacy classes, and has been proved by Geck and Pfeiffer (1.1.2).

The $J$-conjugacy classes partition a double coset $W_{J} d W_{J}$, so for any $J$-conjugacy class $\mathfrak{C}$ we may take an arbitrary element $w=a d b$ for $a, b \in W_{J}$ and $d$ distinguished and $l(w)=l(a)+l(d)+l(b)$. Then to begin with we can conjugate $w$ without increasing length to give $d b a$, with $b a \in W_{J}$.

Suppose $W_{J}^{d} \cap W_{J}=W_{J^{\prime}}$. Then $J^{\prime}$ is a principal subset of $J$ by (4.2.4), and so by our induction hypothesis the group $W_{J}$ has the property that all its $J^{\prime}$-conjugacy classes are reducible ( $W_{J}$ is a Weyl group of type $A_{i}$ or $B_{i}$ for $i \leq k$ ).

Now if $W_{J}^{d} \cap W_{J}=W_{J^{\prime}}$, then the shortest elements of $\mathfrak{C}$ are (by (4.3.1) above) of form $d w$ where $w$ is shortest in its $J^{\prime}$-conjugacy class.

Thus, since $b a$ is reducible in its $J^{\prime}$-conjugacy class to the minimal element $u$, say, and since $s d=d s$ for all $s \in J^{\prime}$ (by (4.2.3)(ii)), the same series of conjugations will reduce $d b a$ to $d u \in \mathfrak{C}_{\text {min }}$.

Thus $\mathfrak{C}$ is reducible in $W_{k+1}$.
The following is a generalization of (1.1.2)(ii).
(4.3.3) Theorem. Let $W$ be of type $A$ or $B$, and let $J \subseteq S$ be principal. If $w$ and $w^{\prime}$ are in $\mathfrak{C}_{\min }$ for some J-conjugacy class $\mathfrak{C}$, then there exists a sequence of $x_{i} \in W_{J}$ and $w_{i} \in \mathfrak{C}_{\min }$ such that $w=w_{0}, x_{i} w_{i} x_{i}^{-1}=w_{i+1}$, and $w_{n+1}=w^{\prime}$, with either $l\left(x_{i} w_{i}\right)=l\left(x_{i}\right)+l\left(w_{i}\right)$ or $l\left(w_{i} x_{i}^{-1}\right)=l\left(w_{i}\right)+l\left(x_{i}^{-1}\right)$ for each $i, 0 \leq i \leq n$.

Proof. We proceed in a similar way to the previous theorem. It is easy to check
for $W_{1}=W\left(A_{1}\right)$, or $W\left(B_{2}\right)$, so we may suppose inductively that it holds for all $W_{i}$ for $i \leq k<n$.

Let us first let $\mathfrak{C}$ be a $J$-conjugacy class, where $|J|<k+1$, and let $w=u d u^{\prime}$ be the reduced expression of any shortest element of $\mathfrak{C}$, where $d$ is distinguished in $W_{J} w W_{J}$. Then $d u^{\prime} u^{-1} \in \mathfrak{C}_{\text {min }}$ also, and $l\left(d u^{\prime} u^{-1}\right)=l\left(d u^{\prime}\right)+l(u)$. In other words, for any element $w=u d u^{\prime}$ of $\mathfrak{C}_{\min }$ we have that there exists a $w^{\prime}=d v \in \mathfrak{C}_{\min }$ for which the claim is satisfied. This reduces the problem to showing that any pair $d v$ and $d v^{\prime}$ in $\mathfrak{C}_{\text {min }}$ have the required property.

Now, suppose $W_{J}^{d} \cap W_{J}=W_{J^{\prime}}$, where $J^{\prime}$ is a principal subset of $J$ (we have shown this is always the case when $J$ is principal in (4.2.4)). Then by (4.3.1), we have $d v \in \mathfrak{C}_{\text {min }}$ implies $v$ is minimal in its $J^{\prime}$-conjugacy class. Since $J^{\prime} \subset S$, we have the result by induction for $v$ and $v^{\prime}$ in $W J$ and the sequence of conjugations by $x_{i}$ all in $W_{J^{\prime}}$. It follows that the result holds for $d v$ and $d v^{\prime}$.

If $J=k+1$, then $J=S$ and this is a special case of (1.1.2)(ii).
We have now shown a complete analogy of (1.1.2) in the case $W$ is of type $A$ or $B$ and $J$ is principal. By the remarks in 3.5 , we now have the following theorem.
(4.3.4) Theorem. The set $Z_{\mathcal{H}}\left(\mathcal{H}_{J}\right)_{\min }^{+}$is an $R$-basis for $Z_{\mathcal{H}}\left(\mathcal{H}_{J}\right)$ if $\mathcal{H}$ is of type $A$ or $B$, and $J$ is principal. Further, the elements $\Gamma_{\mathfrak{C}}$ of $Z_{\mathcal{H}}\left(\mathcal{H}_{J}\right)_{\min }^{+}$are characterized by the properties
(i) $\left.\Gamma_{\mathfrak{C}}\right|_{\xi_{s}=0, s \in S}=\tilde{T}_{\mathfrak{C}}$, and
(ii) $\Gamma_{\mathfrak{C}}-\tilde{T}_{\mathfrak{C}}$ has no shortest elements of any J-conjugacy class.

## Chapter 5 Non-crystallographic types and examples

Theorem (1.1.2) was restricted to Weyl groups, so to extend our results to the non-crystallographic cases it is necessary to prove an analogy for the conjugacy classes in these cases. We deal firstly in this chapter with the case $W$ is dihedral (giving the cases of types $A_{2}$ and $B_{2}$ as examples), and $W$ of type $H_{3}$ is covered in Appendix B.

We also provide a complete proof of the reducibility of $J$-conjugacy classes if $|J|=2$ and $W$ is a Weyl group.

Finally in this chapter, we give some explicit examples of the minimal bases for several centres - in types $A_{3}, A_{4}$, and $B_{3}$.

### 5.1. The dihedral groups

Note that types $A_{2}, B_{2}$, and $G_{2}$ are special cases of the dihedral groups, so this section will deal with them as special cases, giving the minimal bases in types $A_{2}$ and $B_{2}$.

As a consequence of finding the minimal basis of the centre of $\mathcal{H}(I(n))$ below, we have the complete solution for all centralizers of parabolic subalgebras in the dihedral case, since the only non-trivial parabolic subgroups of the dihedral group are generated by a single simple reflection, and so are covered by the theory in section 3.1.

The dihedral group of type $I(n)$ has generators $\{s, t\}$ with relations $s^{2}=t^{2}=$ $(s t)^{n}=1$, frequently represented by the following Dynkin diagram.


The conjugacy classes split into the case where $n$ is even, and the case where $n$ is odd. Let us write $n=2 v$ or $n=2 v+1$.
(5.1.1) Proposition. Every conjugacy class in a dihedral group $I(n)$ is reducible.

Proof. Consider first $n=2 v$. The longest word of $I(n)$ is $(s t)^{v}$. There are $v+3$ conjugacy classes, with representatives $1, s, t,(s t)^{k}$ for $1 \leq k \leq v$. If we write $C_{w}$
for the conjugacy class containing $w \in I(n)$, we have

$$
\begin{aligned}
C_{1} & =\{1\}, \\
C_{s} & =\left\{(s t)^{k} s,(t s)^{l} t \mid k \leq v-2 \text { even, } l \leq v-1 \text { odd }\right\}, \\
C_{t} & =\left\{(s t)^{k} s,(t s)^{l} t \mid k \leq v-1 \text { odd, } l \leq v-1 \text { even }\right\}, \\
C_{(s t)^{k}} & =\left\{(s t)^{k},(t s)^{k}\right\} \quad \text { for } k<v, \\
C_{(s t)^{v}} & =\left\{(s t)^{v}\right\} .
\end{aligned}
$$

The classes $C_{1}$ and $C_{(s t)^{v}}$ are singleton sets, so the theorem trivially holds. The classes $C_{(s t)^{k}}$ for $1 \leq k<v$ have only two elements in each, which are both "minimal" in length in the class, so the proposition holds. Finally, every element of $C_{s}$ and $C_{t}$ has a shorter conjugate by either $s$ or $t$, except the minimal element of the class, so the proposition holds here too.

If $n=2 v+1$, the longest word is $(s t)^{v} s=(t s)^{v} t$, and there are $v+2$ conjugacy classes with representatives $1, s,(s t)^{k}$. The classes are

$$
\begin{aligned}
C_{1} & =\{1\} \\
C_{s} & =\left\{(s t)^{k} s,(t s)^{l} t \mid 0 \leq k \leq v, 0 \leq l<v\right\} \\
C_{(s t)^{k}} & =\left\{(s t)^{k},(t s)^{k}\right\} \quad \text { for } 1 \leq k \leq v
\end{aligned}
$$

Again $C_{1}$ is trivial, and each $C_{(s t)^{k}}$ contains only two elements of the same length. As with the even case, every element of $C_{s}$ has an $s$ - or $t$-conjugate of strictly shorter length, so the proposition holds.

For $\sigma \in S$, we denote the subset of elements in $C_{\sigma}$ of length greater than or equal to $i$ by $C_{\sigma, i}$.

Given (5.1.1), we may use the algorithm $\mathfrak{A}$ to find the primitive minimal positive central elements of the Iwahori-Hecke algebras of the dihedral groups. We provide without proof the following set of elements of $Z(\mathcal{H}(I(n)))_{\min }^{+}$.
(5.1.2) Theorem. The following set of elements is the set of primitive minimal positive central elements of the Iwahori-Hecke algebra of the dihedral group $I(n)$, and thus forms an $R$-basis for $Z(\mathcal{H}(I(n)))$ :

$$
\begin{array}{cr}
n=2 v \text { even, } 1 \leq k \leq v: & n=2 v+1 \text { odd }, 1 \leq k \leq v: \\
\Gamma_{1}=\tilde{T}_{C_{1}}, & \Gamma_{1}=\tilde{T}_{C_{1}}, \\
\Gamma_{\sigma}=\tilde{T}_{C_{\sigma}} \text { for } \sigma \in S, & \Gamma_{s}=\tilde{T}_{C_{s}}, \\
\Gamma_{(s t)^{k}}=\tilde{T}_{C_{(s t)^{k}}}+\sum_{\substack{i>2 k \\
\sigma \in S}} \xi_{\sigma} \tilde{T}_{C_{\sigma, i}} ; & \Gamma_{(s t)^{k}}=\tilde{T}_{C_{(s t)^{k}}}+\xi_{s} \sum_{i>2 k} \tilde{T}_{C_{s, i}} .
\end{array}
$$

Note that in the case $n$ is odd, the generators $s$ and $t$ are conjugate, and so $\xi_{s}=\xi_{t}$.

These elements may be compared with the similar $R$-basis for the centre found by Fakiolas (in [Fa]) working over the ring $\mathbb{Q}[q]$, which we denote $b_{w}$ for $w$ a representative of the conjugacy class $C$. We have the following relations between the elements in [Fa] (modified to be over $R$ ) and those above:

For $n=2 v$ even, $1 \leq k<v: \quad$ For $n=2 v+1$ odd, $1 \leq k \leq v:$

$$
\begin{array}{rlrl}
b_{1} & =\Gamma_{1}, & b_{1} & =\Gamma_{1}, \\
b_{\sigma} & =\Gamma_{\sigma}, & b_{s} & =\Gamma_{s}, \\
b_{(s t)^{k}} & =\Gamma_{(s t)^{k}}-\xi_{s} \Gamma_{s}-\xi_{t} \Gamma_{t}, & b_{(s t)^{k}} & =\Gamma_{(s t)^{k}}-\xi_{s} \Gamma_{s} . \\
b_{(s t)^{v}} & =\Gamma_{(s t)^{v}} . &
\end{array}
$$

Type $A_{2}$.
The Weyl group of type $A_{2}$ is the dihedral group $I(3)$, generated by the simple reflections $s_{1}$ and $s_{2}$ with relations $s_{i}^{2}=\left(s_{1} s_{2}\right)^{3}=1$. It has six elements, $\left\{1, s_{1}, s_{2}, s_{1} s_{2}, s_{2} s_{1}, s_{1} s_{2} s_{1}\right\}$, and three conjugacy classes. Since $s_{1}$ and $s_{2}$ are in the same conjugacy class, we write $\xi_{s_{1}}=\xi_{s_{2}}=\xi$. The conjugacy classes $C_{i}$ of $W\left(A_{2}\right)$ and the corresponding primitive minimal positive central elements are given below:

$$
\begin{array}{lll}
C_{1}=\{1\}, & C_{2}=\left\{s_{1}, s_{2}, s_{1} s_{2} s_{1}\right\}, & C_{3}=\left\{s_{1} s_{2}, s_{2} s_{1}\right\} \\
\Gamma_{1}=\tilde{T}_{C_{1}}, & \Gamma_{2}=\tilde{T}_{C_{2}}, & \Gamma_{3}=\tilde{T}_{C_{3}}+\xi \tilde{T}_{s_{1} s_{2} s_{1}}
\end{array}
$$

## Type $B_{2}$.

The Weyl group of type $B_{2}$ is the dihedral group $I(4)$, and has eight elements generated by the simple reflections $\{s, t\}$ with relations $s^{2}=t^{2}=(s t)^{4}=1$. It
has five conjugacy classes, but the only primitive minimal positive central element which is not simply the conjugacy class sum corresponds to the class $\{s t, t s\}$. We show diagramatically the construction using the algorithm (note that sts commutes with $t$, and $t s t$ with $s$, so that $\tilde{T}_{s t s}\left(\right.$ resp. $\left.\tilde{T}_{t s t}\right)$ is a $t$-class element (resp. $s$-class element) on its own):

5.2 Type $H_{3}$

In this section, we extend (1.1.2) to type $H_{3}$ by explicit calculation of the conjugacy classes. We have not done this for $H_{4}$, as explicit calculation gets a touch difficult in a group of 14400 elements - $H_{3}$ has only 120. However it seems likely that one could use a computer algebra package such as Magma or GAP to do similar calculations to those we have done below for $H_{3}$. On the other hand a more elegant proof may be obtainable using the embedding of $H_{4}$ into $E_{8}$ (see [S]).

Here we write $W=W\left(H_{3}\right)$, with generators $t, s_{1}$, and $s_{2}$, and relations $\left(t s_{1}\right)^{5}=$ $\left(s_{1} s_{2}\right)^{3}=\left(t s_{2}\right)^{2}=s_{i}^{2}=t^{2}=1$ for $i=1,2$. This can be represented by the following Dynkin diagram:


We show in Appendix B the diagrams corresponding to conjugacy classes. The nodes of the diagrams are elements of $W$, and there is an edge joining two nodes when the nodes are conjugate by a simple reflections $s$. The edge joining them then is given the label $s$.

Observation of the diagrams reveals the following analogy of (1.1.2):
(5.2.1) Lemma. Every conjugacy class $C$ in the type $H_{3}$ Coxeter group is reducible. Furthermore, $C_{\min }=C^{w}$ for any $w \in C_{\min }$ in all conjugacy classes
except $C=C_{t}$, in which case $C_{\text {min }}$ is partitioned by three such equivalence classes, $C_{\min }=C^{t} \cup C^{s_{1}} \cup C^{s_{2}}$, with $\left(t s_{1} t s_{1}\right) t\left(s_{1} t s_{1} t\right)=s_{1}$, and $\left(s_{1} s_{2}\right) s_{1}\left(s_{2} s_{1}\right)=s_{2}$, where $l\left(t s_{1} t s_{1} t\right)=l\left(t s_{1} t s_{1}\right)+l(t)$ and $l\left(s_{1} s_{2} s_{1}\right)=l\left(s_{1} s_{2}\right)+l\left(s_{1}\right)$.

By the results of chapter 3, and the remarks in 3.5, we have the following extension of (2.2.1).
(5.2.2) Theorem. Let $\mathcal{H}$ be the Iwahori-Hecke algebra of type $H_{3}$. Then the class elements exist, the algorithm $\mathfrak{A}$ is well-defined on the conjugacy class sum, and the set $Z(\mathcal{H})_{\min }^{+}$is an $R$-basis for $Z(\mathcal{H})$.

### 5.3 The centralizer of two generators

In this section we prove analogous statements to (1.1.2) for the centralizer of a parabolic subalgebra generated by only two elements. That is, if $J \subseteq S$, we are dealing with the case $|J|=2$. The proofs are entirely by a case-by-case demonstration.

We will assume $J=\{s, t\} \subseteq S$, and the order of the product st is 2,3 , or 4 . The case $(s t)^{6}=1$ only occurs in the case $J=S$ in type $G_{2}$, and we omit $(s t)^{5}=1$, which only occurs in types $H_{3}$ and $H_{4}$.

Whatever the order of st, there are at most three cases of the intersection $W_{J}^{d} \cap W_{J}$, which can be either $\{1\},\langle s\rangle$ or $\langle t\rangle$, or $\langle s, t\rangle=W_{J}$. Note that we treat the cases $\langle s\rangle$ and $\langle t\rangle$ as the same, since we have not imposed an order on $s$ and $t$. This approach simply categorizes the double coset representatives into three different cases, which makes them easy to deal with. In theory, this approach can be used for any sized $J$ (we have already used it for $|J|=1$ ), but the cases become impractically numerous.

In each case, we will first list the $\{s, t\}$-conjugacy classes in the double coset $\langle s, t\rangle d\langle s, t\rangle$, and then give the $\{s, t\}$-class element of $Z_{\mathcal{H}}\left(\mathcal{H}_{\{s, t\}}\right)$ corresponding to each $\{s, t\}$-conjugacy class, which has been calculated using the algorithm $\mathfrak{A}$. Occasionally we will represent the calculation by a diagram to illustrate the process.
(5.3.1) $(s t)^{2}=1,\langle s, t\rangle^{d} \cap\langle s, t\rangle=\{1\}$.

The trivial intersection case is the case where there are no relations between $d$ and either $s$ or $t$.

The $\{s, t\}$-conjugacy classes here are:
(1) $\{d, s d s, t d t, s t d t s\}$,
(2) $\{d s, s d, t s d t, t d s t\}$,
(3) $\{d t, t d, s t d s, s d t s\}$,
(4) $\{d s t, t d s, s d t, s t d\}$.

The corresponding $\{s, t\}$-class elements are:
(1) $\tilde{T}_{d}+\tilde{T}_{s d s}+\tilde{T}_{t d t}+\tilde{T}_{s t d t s}$,
(2) $\tilde{T}_{d s}+\tilde{T}_{s d}+\tilde{T}_{t s d t}+\tilde{T}_{t d s t}+\xi\left(\tilde{T}_{s d s}+\tilde{T}_{s t d t s}\right)$,
(3) $\tilde{T}_{d t}+\tilde{T}_{t d}+\tilde{T}_{s t d s}+\tilde{T}_{s d t s}+\xi\left(\tilde{T}_{t d t}+\tilde{T}_{s t d t s}\right)$,
(4) $\tilde{T}_{d s t}+\tilde{T}_{t d s}+\tilde{T}_{s d t}+\tilde{T}_{s t d}+\xi\left(\tilde{T}_{s t d s}+\tilde{T}_{s t d t}+\tilde{T}_{t d s t}+\tilde{T}_{s d s t}\right)+\xi^{2} \tilde{T}_{s t d t s}$.

As an example of the calculation, number (2) can be represented by the diagram:

(5.3.2) $(s t)^{2}=1,\langle s, t\rangle^{d} \cap\langle s, t\rangle=\langle s\rangle$.

In this case, the intersection $\langle s\rangle$ implies that $d^{-1} x d=s$ for some $x \in\{s, t\}$, so we have either (i) $d s=s d$ or (ii) $d s=t d$.
(i) In the case that $d s=s d$ we have the $\{s, t\}$-conjugacy classes:
(1) $\{d, t d t\}$,
(2) $\{d s, t d t s\}$,
(3) $\{d t, t d\}$,
(4) $\{d s t, t d s\}$.

Then the $\{s, t\}$-class elements are
(1) $\tilde{T}_{d}+\tilde{T}_{t d t}$,
(2) $\tilde{T}_{d s}+\tilde{T}_{t d t s}$,
(3) $\tilde{T}_{d t}+\tilde{T}_{t d}+\xi \tilde{T}_{t d t}$,
(4) $\tilde{T}_{d s t}+\tilde{T}_{t d s}+\xi \tilde{T}_{t d t s}$.
(ii) In the case $d s=t d$ we have the $\{s, t\}$-conjugacy classes:
(1) $\{d, d s t, s d s, s d t\}$,
(2) $\{s d, d s, d t, s d s t\}$.

Then the $\{s, t\}$-class elements are:
(1) $\tilde{T}_{d}+\tilde{T}_{d s t}+\tilde{T}_{s d s}+\tilde{T}_{s d t}+\xi \tilde{T}_{s d s t}$,
(2) $\tilde{T}_{s d}+\tilde{T}_{d s}+\tilde{T}_{d t}+\tilde{T}_{s d s t}+\xi\left(\tilde{T}_{s d s}+\tilde{T}_{d s t}+\tilde{T}_{s d t}\right)+\xi^{2} \tilde{T}_{s d s t}$.
(5.3.3) $(s t)^{2}=1,\langle s, t\rangle^{d} \cap\langle s, t\rangle=\langle s, t\rangle$.

Here we also have two cases. Either (i) $s d=d s$ and $t d=d t$ or (ii) $s d=d t$ and $t d=d s$.
(i) If $s d=d s$ and $t d=d t$ the $\{s, t\}$-conjugacy classes are singleton sets: $\{d\}$, $\{d s\},\{d t\}$, and $\{d t s\}$, and the $\{s, t\}$-class elements are the single generators $\tilde{T}_{d}$, $\tilde{T}_{d s}, \tilde{T}_{d t}$, and $\tilde{T}_{d t s}$.
(ii) If $s d=d t$ and $t d=d s$ there are two $\{s, t\}$-conjugacy classes $\{d, d s t\}$ and $\{d t, d s\}$, and their corresponding $\{s, t\}$-class elements are $\tilde{T}_{d}+\tilde{T}_{d s t}$ and $\tilde{T}_{d s}+\tilde{T}_{d t}+$ $\xi \tilde{T}_{d t s}$.
(5.3.4) $(s t)^{3}=1,\langle s, t\rangle^{d} \cap\langle s, t\rangle=\{1\}$.

Here there are six $\{s, t\}$-conjugacy classes in the double coset:
(1) $\{d, s d s, t d t, t s d s t, s t d t s$, stsdsts $\}$,
(2) $\{d s, s d, t d s t, t s d t, s t d s t s, s t s d t s\}$,
(3) $\{d t, t d, s d t s, s t d s, t s d s t s, s t s d s t\}$,
(4) $\{d t s, s d t, t s d, t d s t s, s t s d s, s t d s t\}$,
(5) $\{d s t, t d s, s t d, s d s t s, s t s d t, t s d t s\}$,
(6) $\{d s t s, s d s t, t d t s, t s d s, s t d t, s t s d\}$.

Clearly (2) and (3) are equivalent and (4) and (5) are equivalent, under swapping $s$ and $t$. The corresponding $\{s, t\}$-class elements (ignoring equivalent ones) are:
(1) $\tilde{T}_{d}+\tilde{T}_{s d s}+\tilde{T}_{t d t}+\tilde{T}_{s t d t s}+\tilde{T}_{t s d s t}+\tilde{T}_{s t s d s t s}$,
(2) $\tilde{T}_{d s}+\tilde{T}_{s d}+\tilde{T}_{t d s t}+\tilde{T}_{t s d t}+\tilde{T}_{s t d s t s}+\tilde{T}_{s t s d t s}+\xi\left(\tilde{T}_{s d s}+\tilde{T}_{t s d s t}+\tilde{T}_{s t s d s t s}\right)$,
(4) $\tilde{T}_{d t s}+\tilde{T}_{s d t}+\tilde{T}_{t s d}+\tilde{T}_{t d s t s}+\tilde{T}_{s t d s t}+\tilde{T}_{s t s d s}+\xi\left(\tilde{T}_{s d t s}+\tilde{T}_{t s d t}+\tilde{T}_{s t d s t s}+\right.$ $\left.\tilde{T}_{s t s d t s}+\tilde{T}_{t s d s t s}+\tilde{T}_{s t s d s t}\right)+\xi^{2} \tilde{T}_{s t s d s t s}$,
(6) $\tilde{T}_{d s t s}+\tilde{T}_{s d s t}+\tilde{T}_{t d t s}+\tilde{T}_{t s d s}+\tilde{T}_{s t d t}+\tilde{T}_{s t s d}+\xi\left(\tilde{T}_{t d s t s}+\tilde{T}_{s t d t s}+\tilde{T}_{s t d s t}+\right.$ $\left.\tilde{T}_{s t s d s}+\tilde{T}_{t s d s t}+\tilde{T}_{t s d t s}+\tilde{T}_{s t s d t}+\tilde{T}_{s t s d s t s}\right)+\xi^{2}\left(\tilde{T}_{s t d s t s}+\tilde{T}_{s t s d t s}+\tilde{T}_{t s d s t s}+\right.$ $\left.\tilde{T}_{s t s d s t}\right)+\xi^{3} \tilde{T}_{s t s d s t s}$.

We will do (4) as a diagrammatical example:

(5.3.5) $(s t)^{3}=1,\langle s, t\rangle^{d} \cap\langle s, t\rangle=\langle s\rangle$.

Again, two cases: (i) $d s=s d$; and (ii) $d s=t d$.
(i) The $\{s, t\}$-conjugacy classes are
(1) $\{d, t d t, s t d t s\}$,
(2) $\{d s, t d s t, s t d s t s\}$,
(3) $\{t d, d t, s t s d, s t d t, s d t s, t d t s\}$,
(4) $\{d t s, d s t, t d s, s t d, s t s d t, t d s t s\}$.

The corresponding $\{s, t\}$-class elements are:
(1) $\tilde{T}_{d}+\tilde{T}_{t d t}+\tilde{T}_{s t d t s}$,
(2) $\tilde{T}_{d s}+\tilde{T}_{t d s t}+\tilde{T}_{s t d s t s}$,
(3) $\tilde{T}_{d t}+\tilde{T}_{t d}+\tilde{T}_{s d t s}+\tilde{T}_{s t s d}+\tilde{T}_{s t d t}+\tilde{T}_{t d t s}+\xi\left(\tilde{T}_{t d t}+\tilde{T}_{s t d s t}+\tilde{T}_{t s d t s}+2 \tilde{T}_{s t d t s}\right)+$ $\xi^{2} \tilde{T}_{s t d s t s}$,
(4) $\tilde{T}_{d t s}+\tilde{T}_{s d t}+\tilde{T}_{t s d}+\tilde{T}_{s t d}+\tilde{T}_{t d s t s}+\tilde{T}_{s t d s t}+\xi\left(\tilde{T}_{s d t s}+\tilde{T}_{t d s t}+\tilde{T}_{t d t s}+\tilde{T}_{s t s d}+\right.$ $\left.\tilde{T}_{s t d t}+2 \tilde{T}_{s t d s t s}\right)+\xi^{2}\left(\tilde{T}_{t d s t s}+\tilde{T}_{s t d s t}+\tilde{T}_{s t d t s}\right)+\xi^{3} \tilde{T}_{s t d s t s}$.
(ii) The $\{s, t\}$-conjugacy classes are:
(1) $\{d, s d s, d s t, t s d s t, t s d t s, s d s t s\}$,
(2) $\{d s, s d, d t, t s d t, s d t s, t s d s t s\}$,
(3) $\{d t s, s d t, t s d\}$,
(4) $\{d s t s, s d s t, t s d s\}$.

The corresponding $\{s, t\}$-class elements are:
(1) $\tilde{T}_{d}+\tilde{T}_{d s t}+\tilde{T}_{s d s}+\tilde{T}_{t s d s t}+\tilde{T}_{s d s t s}+\tilde{T}_{t s d t s}+\xi \tilde{T}_{t s d s t s}$,
(2) $\tilde{T}_{d s}+\tilde{T}_{s d}+\tilde{T}_{d t}+\tilde{T}_{s d t s}+\tilde{T}_{t s d t}+\tilde{T}_{t s d s t s}+\xi\left(\tilde{T}_{s d s}+\tilde{T}_{d s t}+\tilde{T}_{s d s t s}+\tilde{T}_{t s d s t}+\right.$ $\left.\tilde{T}_{t s d t s}\right)+\xi^{2} \tilde{T}_{t s d s t s}$,
(3) $\tilde{T}_{d t s}+\tilde{T}_{s d t}+\tilde{T}_{t s d}+\xi\left(\tilde{T}_{t s d t}+\tilde{T}_{s d t s}+\tilde{T}_{t s d s t s}\right)$,
(4) $\tilde{T}_{d s t s}+\tilde{T}_{s d s t}+\tilde{T}_{t s d s}+\xi\left(\tilde{T}_{t s d s t}+\tilde{T}_{t s d t s}+\tilde{T}_{s d s t s}\right)+\xi^{2} \tilde{T}_{t s d s t s}$.

The following is the diagramatical construction of (4):


Note here that $\tilde{T}_{d s t s}$ is a $t$-class element on its own, and $\tilde{T}_{t s d s}$ is an $s$-class element on its own.
(5.3.6) $(s t)^{3}=1,\langle s, t\rangle^{d} \cap\langle s, t\rangle=\langle s, t\rangle$.

Two cases again: (i) $s d=d s$ and $t d=d t$; and (ii) $s d=d t$ and $t d=d s$.
(i) The $\{s, t\}$-conjugacy classes are:
(1) $\{d\}$,
(2) $\{d s, d t, d s t s\}$,
(3) $\{d t s, d s t\}$.

The corresponding $\{s, t\}$-class elements are:
(1) $\tilde{T}_{d}$,
(2) $\tilde{T}_{d t}+\tilde{T}_{d s}+\tilde{T}_{d s t s}$,
(3) $\tilde{T}_{d t s}+\tilde{T}_{d s t}+\xi \tilde{T}_{d s t s}$.
(ii) The $\{s, t\}$-conjugacy classes are:
(1) $\{d, d t s, d s t\}$,
(2) $\{d s, d t\}$,
(3) $\{d s t s\}$.

The corresponding $\{s, t\}$-class elements are:
(1) $\tilde{T}_{d}+\tilde{T}_{d t s}+\tilde{T}_{d s t}$,
(2) $\tilde{T}_{d t}+\tilde{T}_{d s}+\xi\left(\tilde{T}_{d t s}+\tilde{T}_{d s t}\right)$,
(3) $\tilde{T}_{d s t s}$.
(5.3.7) $(s t)^{4}=1,\langle s, t\rangle^{d} \cap\langle s, t\rangle=\{1\}$.

There are eight $\{s, t\}$-conjugacy classes:
(1) $\{d, s d s, t d t, t s d s t, s t d t s, s t s d s t s, t s t d t s t$, ststdtsts $\}$,
(2) $\{d s, s d, t d s t, t s d t, s t d s t s$, stsdts, tstdtsts, ststdtst\},
(3) $\{d t, t d, s d t s, s t d s, t s d t s t, t s t d s t, s t s d t s t s, s t s t d s t s\}$,
(4) $\{d s t, t d s, s t d, s d s t s, t s t d t, t s d t s t s$, ststdts, stsdtst\},
(5) $\{d t s, s d t, t s d, t d t s t, s t s d s, s t d t s t s$, ststdst, $t s t d s t s\}$,
(6) $\{d s t s, s d s t, t s d s$, stsd, tdtsts, ststdt, stdtst, tstdts\},
(7) $\{d t s t, t d t s, s t d t, t s t d, s d t s t s, s t s t d s, t s d s t s, s t s d s t\}$,
(8) $\{d t s t s, s d t s t, t s d t s, s t s d t$, ststd, $t d s t s$, stdst, tstds\}.

Note that here, as in the $(s t)^{3}=1$ case, we have some symmetry between classes (2) and (3), (4) and (5), and (6) and (7). Thus we will omit the obvious symmetric cases in the $\{s, t\}$-class elements below:
(1) $\tilde{T}_{d}+\tilde{T}_{s d s}+\tilde{T}_{t d t}+\tilde{T}_{t s d s t}+\tilde{T}_{s t d t s}+\tilde{T}_{s t s d s t s}+\tilde{T}_{t s t d t s t}+\tilde{T}_{s t s t d t s t s}$,
(2) $\tilde{T}_{d s}+\tilde{T}_{s d}+\tilde{T}_{t d s t}+\tilde{T}_{t s d t}+\tilde{T}_{s t d s t s}+\tilde{T}_{s t s d t s}+\tilde{T}_{t s t d t s t s}+\tilde{T}_{s t s t d t s t}+\xi\left(\tilde{T}_{s d s}+\right.$ $\left.\tilde{T}_{t s d s t}+\tilde{T}_{s t s d s t s}+\tilde{T}_{s t s t d t s t s}\right)$,
(4) $\tilde{T}_{d s t}+\tilde{T}_{t d s}+\tilde{T}_{s t d}+\tilde{T}_{s d s t s}+\tilde{T}_{t s t d t}+\tilde{T}_{t s d t s t s}+\tilde{T}_{s t s t d t s}+\tilde{T}_{s t s d t s t}+\xi\left(\tilde{T}_{t d s t}+\tilde{T}_{s t d s}+\right.$ $\left.\tilde{T}_{s t d s t s}+\tilde{T}_{t s t d s t}+\tilde{T}_{t s t d t s t s}+\tilde{T}_{s t s d t s t s}+\tilde{T}_{s t s t d t s t}+\tilde{T}_{s t s t d s t s}\right)+\xi^{2} \tilde{T}_{s t s t d t s t s}$,
(6) $\tilde{T}_{d s t s}+\tilde{T}_{s d s t}+\tilde{T}_{t s d s}+\tilde{T}_{s t s d}+\tilde{T}_{t d t s t s}+\tilde{T}_{s t s t d t}+\tilde{T}_{s t d t s t}+\tilde{T}_{t s t d t s}+\xi\left(\tilde{T}_{s d s t s}+\right.$ $\tilde{T}_{t s d s t}+\tilde{T}_{s t s d s}+\tilde{T}_{t s t d t s t}+\tilde{T}_{t s t d s t s}+\tilde{T}_{s t d t s t s}+\tilde{T}_{s t s t d s t}+\tilde{T}_{s t s d s t s}+\tilde{T}_{t s d t s t s}+$ $\left.\tilde{T}_{s t s d t s t}+\tilde{T}_{s t s t d t s}+\tilde{T}_{s t s t d t s t s}\right)+\xi^{2}\left(\tilde{T}_{t s t d t s t s}+\tilde{T}_{s t s d t s t s}+\tilde{T}_{s t s t d t s t}+\tilde{T}_{s t s t d s t s}\right)+$ $\xi^{3} \tilde{T}_{\text {ststdtsts }}$,
(8) $\tilde{T}_{d t s t s}+\tilde{T}_{s d t s t}+\tilde{T}_{t s d t s}+\tilde{T}_{s t s d t}+\tilde{T}_{s t s t d}+\tilde{T}_{t d s t s}+\tilde{T}_{s t d s t}+\tilde{T}_{t s t d s}+\xi\left(\tilde{T}_{s d t s t s}+\right.$ $\tilde{T}_{t s d s t s}+\tilde{T}_{t s d t s t}+\tilde{T}_{t d t s t s}+\tilde{T}_{s t d s t s}+\tilde{T}_{s t d t s t}+\tilde{T}_{s t s d t s}+\tilde{T}_{s t s d s t}+\tilde{T}_{t s t d s t}+$ $\left.\tilde{T}_{t s t d t s}+\tilde{T}_{s t s t d t}+\tilde{T}_{s t s t d s}+\tilde{T}_{s t s d t s t s}+\tilde{T}_{t s t d t s t s}+\tilde{T}_{s t s t d t s t}+\tilde{T}_{s t s t d s t s}\right)+$ $\xi^{2}\left(\tilde{T}_{t s d t s t s}+\tilde{T}_{s t d t s t s}+\tilde{T}_{s t s d s t s}+\tilde{T}_{s t s d t s t}+\tilde{T}_{t s t d t s t}+\tilde{T}_{t s t d s t s}+\tilde{T}_{s t s t d s t}+\tilde{T}_{s t s t d t s}+\right.$ $\left.2 \tilde{T}_{s t s t d t s t s}\right)+\xi^{3}\left(\tilde{T}_{s t s d t s t s}+\tilde{T}_{t s t d t s t s}+\tilde{T}_{\text {ststdtst }}+\tilde{T}_{s t s t d s t s}\right)+\xi^{4} \tilde{T}_{s t s t d t s t s}$.
(5.3.8) $(s t)^{4}=1,\langle s, t\rangle^{d} \cap\langle s, t\rangle=\langle s\rangle$.

As with the above, there are two cases: (i) $d s=s d$, and (ii) $d s=t d$.
(i) The $\{s, t\}$-conjugacy classes are
(1) $\{d, t d t$, stdts, tstdtst $\}$,
(2) $\{d s, t d s t, s t d s t s, t s t d t s t s\}$,
(3) $\{d t, t d, d s t s$, stsd, tdtsts, stdtst, tstdts, tstdst $\}$,
(4) $\{d s t, t d s, s t d, d t s, t d t s t, t s t d t, s t d t s t s, t s t d s t s\}$,
(5) $\{d t s t, t d t s, s t d t, t s t d\}$,
(6) $\{d t s t s, t d s t s, s t d s t, t s t d s\}$.

The corresponding $\{s, t\}$-class elements are:
(1) $\tilde{T}_{d}+\tilde{T}_{t d t}+\tilde{T}_{s t d t s}+\tilde{T}_{t s t d t s t}$,
(2) $\tilde{T}_{d s}+\tilde{T}_{t d s t}+\tilde{T}_{s t d s t s}+\tilde{T}_{t s t d t s t s}$,
(3) $\tilde{T}_{d t}+\tilde{T}_{t d}+\tilde{T}_{d s t s}+\tilde{T}_{s t s d}+\tilde{T}_{t d t s t s}+\tilde{T}_{s t d t s t}+\tilde{T}_{t s t d t s}+\tilde{T}_{t s t d s t}+\xi\left(\tilde{T}_{t d t}+\tilde{T}_{s t d t s}+\right.$ $\left.2 \tilde{T}_{t s t d t s t}+\tilde{T}_{s t d t s t s}+\tilde{T}_{t s t d s t s}\right)+\xi^{2} \tilde{T}_{t s t d t s t s}$,
(4) $\tilde{T}_{d s t}+\tilde{T}_{t d s}+\tilde{T}_{s t d}+\tilde{T}_{d t s}+\tilde{T}_{t d t s t}+\tilde{T}_{t s t d t}+\tilde{T}_{s t d t s t s}+\tilde{T}_{t s t d s t s}+\xi\left(\tilde{T}_{d s t s}+\right.$ $\left.\tilde{T}_{t d s t}+\tilde{T}_{s t d s}+\tilde{T}_{t d t s t s}+\tilde{T}_{s t d s t s}+\tilde{T}_{s t d t s t}+\tilde{T}_{t s t d s t}+\tilde{T}_{t s t d t s}+2 \tilde{T}_{t s t d t s t s}\right)+$ $\xi^{2}\left(\tilde{T}_{s t d t s t s}+\tilde{T}_{t s t d t s t}+\tilde{T}_{t s t d s t s}\right)+\xi^{3} \tilde{T}_{t s t d t s t s}$,
(5) $\tilde{T}_{d t s t}+\tilde{T}_{t d t s}+\tilde{T}_{s t d t}+\tilde{T}_{t s t d}+\xi\left(\tilde{T}_{t d t s t}+\tilde{T}_{s t d t s}+\tilde{T}_{t s t d t}+\tilde{T}_{s t d t s t s}+\tilde{T}_{t s t d t s t}+\right.$ $\left.\tilde{T}_{t s t d s t s}\right)+\xi^{2} \tilde{T}_{t s t d t s t s}$,
(6) $\tilde{T}_{d t s t s}+\tilde{T}_{t d s t s}+\tilde{T}_{s t d s t}+\tilde{T}_{t s t d s}+\xi\left(\tilde{T}_{t d t s t s}+\tilde{T}_{s t d s t s}+\tilde{T}_{t s t d s t}+\tilde{T}_{s t d t s t}+\tilde{T}_{t s t d t s}+\right.$ $\left.\tilde{T}_{t s t d t s t s}\right)+\xi^{2}\left(\tilde{T}_{s t d t s t s}+\tilde{T}_{t s t d t s t}+\tilde{T}_{t s t d s t s}\right)+\xi^{3} \tilde{T}_{t s t d t s t s}$.

The following is the diagramatical representation of the construction of (6) above, noting we omit some horizontal lines for convenience:

(ii) The $\{s, t\}$-conjugacy classes are:
(1) $\{d, s d s, d s t, t s d s t, s d s t s, t s d t s t s$, stsdsts, stsdtst $\}$,
(2) $\{d s, s d, d t, s d t s, t s d t, t s d t s t$, stsdts, stsdtsts\},
(3) $\{d t s, s d t, t s d, d t s t s$, stsds, sdtst, tsdts, stsdt $\}$,
(4) $\{d s t s, s d s t, d t s t, t s d s, s t s d, s d t s t s, t s d s t s, s t s d s t\}$.

The corresponding $\{s, t\}$-class elements are:
(1) $\tilde{T}_{d}+\tilde{T}_{s d s}+\tilde{T}_{d s t}+\tilde{T}_{t s d s t}+\tilde{T}_{s d s t s}+\tilde{T}_{t s d t s t s}+\tilde{T}_{s t s d s t s}+\tilde{T}_{s t s d t s t}+\xi \tilde{T}_{s t s d t s t s}$,
(2) $\tilde{T}_{d s}+\tilde{T}_{s d}+\tilde{T}_{d t}+\tilde{T}_{s d t s}+\tilde{T}_{t s d t}+\tilde{T}_{t s d t s t}+\tilde{T}_{s t s d t s}+\tilde{T}_{s t s d t s t s}+\xi\left(\tilde{T}_{d s t}+\tilde{T}_{s d s}+\right.$ $\left.\tilde{T}_{t s d s t}+\tilde{T}_{s d s t s}+\tilde{T}_{s t s d s t s}+\tilde{T}_{t s d t s t s}+\tilde{T}_{s t s d t s t}\right)+\xi^{2} \tilde{T}_{s t s d t s t s}$,
(3) $\tilde{T}_{d t s}+\tilde{T}_{s d t}+\tilde{T}_{t s d}+\tilde{T}_{d t s t s}+\tilde{T}_{s t s d s}+\tilde{T}_{s d t s t}+\tilde{T}_{t s d t s}+\tilde{T}_{s t s d t}+\xi\left(\tilde{T}_{s d t s}+\tilde{T}_{t s d t}+\right.$ $\left.2 \tilde{T}_{t s d t s t}+\tilde{T}_{\text {sdtsts }}+2 \tilde{T}_{\text {stsdts }}+\tilde{T}_{t s d s t s}+\tilde{T}_{\text {stsdst }}+2 \tilde{T}_{\text {stsdtsts }}\right)+\xi^{2}\left(\tilde{T}_{t s d t s t s}+\right.$ $\left.\tilde{T}_{s t s d t s t}+\tilde{T}_{\text {stsdsts }}\right)+\xi^{3} \tilde{T}_{s t s d t s t s}$,
(4) $\tilde{T}_{d s t s}+\tilde{T}_{s d s t}+\tilde{T}_{d t s t}+\tilde{T}_{t s d s}+\tilde{T}_{s t s d}+\tilde{T}_{s d t s t s}+\tilde{T}_{t s d s t s}+\tilde{T}_{s t s d s t}+\xi\left(\tilde{T}_{t s d t}+\tilde{T}_{s d t s}+\right.$ $\left.2 \tilde{T}_{t s d t s t}+2 \tilde{T}_{\text {stsdts }}+\tilde{T}_{\text {stsdst }}+\tilde{T}_{t s d s t s}+\tilde{T}_{\text {sdtsts }}+2 \tilde{T}_{\text {stsdtsts }}\right)+\xi^{2}\left(\tilde{T}_{\text {stsdsts }}+\right.$ $\left.\tilde{T}_{t s d t s t s}+\tilde{T}_{s t s d t s t}\right)+\xi^{3} \tilde{T}_{\text {stsdtsts }}$.
(5.3.9) $(s t)^{4}=1,\langle s, t\rangle^{d} \cap\langle s, t\rangle=\langle s, t\rangle$.

Either (i) $d s=s d$ and $d t=t d$ or (ii) $d s=t d$ and $d t=s d$.
(i) The $\{s, t\}$-conjugacy classes are:
(1) $\{d\}$,
(2) $\{d s, d t s t\}$,
(3) $\{d t, d s t s\}$,
(4) $\{d t s, d s t\}$,
(5) $\{d t s t s\}$.

Omitting case (3) which is symmetric to (2) under swapping $s$ and $t$, we have that the corresponding $\{s, t\}$-class elements are:
(1) $\tilde{T}_{d}$,
(2) $\tilde{T}_{d s}+\tilde{T}_{d t s t}$,
(4) $\tilde{T}_{d t s}+\tilde{T}_{d s t}+\xi\left(\tilde{T}_{s t s}+\tilde{T}_{t s t}\right)$,
(5) $\tilde{T}_{d t s t s}$.
(ii) The $\{s, t\}$-conjugacy classes are:
(1) $\{d, d t s, d s t, d t s t s\}$,
(2) $\{d s, d t\}$,
(3) $\{d s t s, d t s t\}$.

The corresponding $\{s, t\}$-class elements are:
(1) $\tilde{T}_{d}+\tilde{T}_{d t s}+\tilde{T}_{d s t}+\tilde{T}_{d t s t s}$,
(2) $\tilde{T}_{d s}+\tilde{T}_{d t}+\xi\left(\tilde{T}_{d t s}+\tilde{T}_{d s t}+\tilde{T}_{d t s t s}\right)$,
(3) $\tilde{T}_{d s t s}+\tilde{T}_{d t s t}+\xi \tilde{T}_{d t s t s}$.

### 5.4 Types $A_{3}$ and $A_{4}$

Type $A_{3}$.
Let $W$ be the Weyl group of type $A_{3}$, generated by $s_{1}, s_{2}$, and $s_{3}$, with relations $s_{j}^{2}=\left(s_{1} s_{3}\right)^{2}=\left(s_{i} s_{i+1}\right)^{3}=1$ for $j=1,2,3$ and $i=1,2$. The conjugacy classes of
$W$ are $C_{i d}=\{1\}$,

$$
\begin{aligned}
C_{12} & =\left\{s_{1} s_{2}, s_{2} s_{1}, s_{2} s_{3}, s_{3} s_{2}, s_{2} s_{3} s_{2} s_{1}, s_{1} s_{2} s_{3} s_{2}, s_{1} s_{2} s_{1} s_{3}, s_{1} s_{3} s_{2} s_{1}\right\} \\
C_{1} & =\left\{s_{1}, s_{2}, s_{3}, s_{1} s_{2} s_{1}, s_{2} s_{3} s_{2}, s_{1} s_{2} s_{3} s_{2} s_{1}\right\} \\
C_{13} & =\left\{s_{1} s_{3}, s_{2} s_{1} s_{3} s_{2}, s_{1} s_{2} s_{1} s_{3} s_{2} s_{1}\right\} \\
C_{123} & =\left\{s_{1} s_{2} s_{3}, s_{2} s_{1} s_{3}, s_{1} s_{3} s_{2}, s_{3} s_{2} s_{1}, s_{1} s_{2} s_{1} s_{3} s_{2}, s_{2} s_{1} s_{3} s_{2} s_{1}\right\} .
\end{aligned}
$$

For simplicity of notation, we will write $\tilde{T}_{s_{i}}=\tilde{T}_{i}$, and as before we will write $\tilde{T}_{C}$ for the conjugacy class sum. Note again (as with $A_{2}$, or any type $A$ Weyl group) that the simple reflections are all in the same conjugacy class, and so the indeterminates $\xi_{s}$ are all equal, and we write them all $\xi$, giving us $R=\mathbb{Z}[\xi]$.

The elements of $Z\left(\mathcal{H}\left(A_{3}\right)\right)_{\min }^{+}$are:

$$
\begin{aligned}
\Gamma_{C}= & \tilde{T}_{C} \quad \text { for } C=C_{i d}, C_{1}, C_{13} \\
\Gamma_{C_{12}}= & \tilde{T}_{C_{12}}+\xi\left(\tilde{T}_{121}+\tilde{T}_{232}+2 \tilde{T}_{12321}+\tilde{T}_{21321}+\tilde{T}_{12132}\right)+\xi^{2} \tilde{T}_{121321}, \\
\Gamma_{C_{123}}= & \tilde{T}_{C_{123}}+\xi\left(\tilde{T}_{1213}+\tilde{T}_{1321}+\tilde{T}_{1232}+\tilde{T}_{2321}+\tilde{T}_{2132}+2 \tilde{T}_{121321}\right)+ \\
& \xi^{2}\left(\tilde{T}_{12132}+\tilde{T}_{21321}+\tilde{T}_{12321}\right)+\xi^{3} \tilde{T}_{121321} .
\end{aligned}
$$

We may compare these elements with the Jones elements (see section 5), which we denote $b_{C}$ for the element corresponding to the conjugacy class $C$ :

$$
\begin{aligned}
b_{C_{i d}} & =24 \Gamma_{C_{i d}}+12 \xi \Gamma_{C_{1}}+6 \xi^{2} \Gamma_{C_{13}}+4 \xi^{2} \Gamma_{C_{12}}+\xi^{3} \Gamma_{C_{123}} \\
b_{C_{1}} & =2 \Gamma_{C_{1}}+2 \xi \Gamma_{C_{12}}+2 \xi \Gamma_{C_{13}}+\xi^{2} \Gamma_{C_{123}}, \\
b_{C_{13}} & =2 \Gamma_{C_{13}}+\xi \Gamma_{C_{123}}, \\
b_{C_{12}} & =\Gamma_{C_{12}}+\xi \Gamma_{C_{123}}, \\
b_{C_{123}} & =\Gamma_{C_{123}} .
\end{aligned}
$$

The upper-triangularity of these relationships reflects the fact that apart from those in $C$ the Jones element corresponding to $C$ contains shortest elements only of conjugacy classes of length greater than $l_{C}$.

We can graphically show the construction of $\Gamma_{C_{12}}$ in $\mathcal{H}\left(A_{3}\right)$, starting at the top with the shortest elements in $C_{12}$, and $s$-class element completions denoted by connecting lines. The practical process is to start with the shortest, and check that for each $s$ there are lines labled by $s$ connecting the element with the others
in its $s$-class element. The shortest term for which there is no connection for some $s$ is the term we complete.

Note that both $\xi \tilde{T}_{12321}$ and $\xi^{2} \tilde{T}_{121321}$ are self-conjugate under $\tilde{T}_{2}$, so form $s_{2^{-}}$ class elements of Type I, and as before, we omit several horizontal lines which are inferred by the presence of type $b_{d s}^{I I} s$-class elements.


One can see that for any $s \in S$, we may cut up the above graph into disjoint subgraphs corresponding to the types shown in Figure (3.1.6) as well as the singleton subgraphs corresponding to type I $s$-class elements, although in the above diagram we have suppressed any horizontal lines from the type II graphs for simplicity. This shows that the sum of the terms above is in the centre, and by checking each step never adds shorter we have (by (3.3.6)) that this sum is the element $\Gamma_{C_{12}} \in Z\left(\mathcal{H}\left(A_{3}\right)\right)_{\min }^{+}$. Alternatively, to see the sum is $\Gamma_{C_{12}}$ one could observe that it specializes to $\tilde{T}_{C_{12}}$ and that there are no shortest elements from any conjugacy class other than those from $\tilde{T}_{C_{12}}$. We can then make our conclusion using the characterization of (3.2.6).

Type $A_{4}$.
The conjugacy classes in type $A_{4}$ correspond to the seven partitions of 5 (see [C1]), giving us representatives $s_{1} s_{2} s_{3} s_{4}, s_{1} s_{2} s_{3}, s_{1} s_{2} s_{4}, s_{1} s_{2}, s_{1} s_{3}, s_{1}$, and 1. The
conjugacy classes are:

$$
\begin{aligned}
& C_{1}=\{1\} \\
& C_{2}=\left\{s_{1}, s_{2}, s_{3}, s_{4}, s_{1} s_{2} s_{1}, s_{2} s_{3} s_{2}, s_{3} s_{4} s_{3}, s_{1} s_{2} s_{3} s_{2} s_{1}, s_{2} s_{3} s_{4} s_{3} s_{2}, s_{1} s_{2} s_{3} s_{4} s_{3} s_{2} s_{1}\right\}, \\
& C_{3}=\left\{s_{1} s_{3}, s_{1} s_{4}, s_{2} s_{4}, s_{2} s_{1} s_{3} s_{2}, s_{1} s_{3} s_{4} s_{3}, s_{1} s_{2} s_{1} s_{4}, s_{3} s_{2} s_{4} s_{3}, s_{1} s_{2} s_{1} s_{3} s_{2} s_{1},\right. \\
& s_{2} s_{1} s_{3} s_{4} s_{3} s_{2}, s_{1} s_{3} s_{2} s_{1} s_{4} s_{3}, s_{2} s_{3} s_{2} s_{4} s_{3} s_{2}, s_{1} s_{2} s_{1} s_{3} s_{4} s_{3} s_{2} s_{1} \\
& \left.s_{2} s_{1} s_{3} s_{2} s_{1} s_{4} s_{3} s_{2}, s_{1} s_{2} s_{3} s_{2} s_{4} s_{3} s_{2} s_{1}, s_{1} s_{2} s_{1} s_{3} s_{2} s_{1} s_{4} s_{3} s_{2} s_{1}\right\} \\
& C_{4}=\left\{s_{1} s_{2}, s_{2} s_{1}, s_{2} s_{3}, s_{3} s_{2}, s_{3} s_{4}, s_{4} s_{3}, s_{1} s_{2} s_{1} s_{3}, s_{3} s_{1} s_{2} s_{1}, s_{1} s_{2} s_{3} s_{2}, s_{2} s_{3} s_{2} s_{1},\right. \\
& s_{2} s_{3} s_{2} s_{4}, s_{2} s_{4} s_{3} s_{2}, s_{2} s_{3} s_{4} s_{3}, s_{3} s_{4} s_{3} s_{2}, s_{1} s_{2} s_{3} s_{4} s_{3} s_{2}, s_{1} s_{2} s_{1} s_{3} s_{4} s_{3}, \\
& \left.s_{1} s_{3} s_{2} s_{1} s_{4} s_{3}, s_{1} s_{2} s_{4} s_{3} s_{2} s_{1}, s_{2} s_{3} s_{4} s_{3} s_{2} s_{1}, s_{1} s_{2} s_{3} s_{2} s_{1} s_{4}\right\} \\
& C_{5}=\left\{s_{1} s_{2} s_{4}, s_{2} s_{1} s_{4}, s_{1} s_{3} s_{4}, s_{1} s_{4} s_{3}, s_{1} s_{3} s_{2} s_{4} s_{3}, s_{3} s_{2} s_{1} s_{4} s_{3}, s_{2} s_{1} s_{3} s_{2} s_{4},\right. \\
& s_{2} s_{1} s_{4} s_{3} s_{2}, s_{2} s_{1} s_{3} s_{2} s_{4} s_{3} s_{2}, s_{1} s_{2} s_{3} s_{2} s_{4} s_{3} s_{2}, s_{2} s_{3} s_{2} s_{4} s_{3} s_{2} s_{1}, s_{2} s_{3} s_{2} s_{1} s_{4} s_{3} s_{2}, \\
& s_{2} s_{1} s_{3} s_{2} s_{1} s_{4} s_{3}, s_{1} s_{2} s_{1} s_{3} s_{2} s_{1} s_{4}, s_{1} s_{2} s_{1} s_{4} s_{3} s_{2} s_{1}, s_{1} s_{3} s_{2} s_{1} s_{4} s_{3} s_{2}, \\
& \left.s_{3} s_{2} s_{1} s_{4} s_{3} s_{2} s_{1}, s_{1} s_{2} s_{1} s_{3} s_{2} s_{4} s_{3}, s_{1} s_{2} s_{3} s_{2} s_{1} s_{4} s_{3} s_{2} s_{1}, s_{1} s_{2} s_{1} s_{3} s_{2} s_{4} s_{3} s_{2} s_{1}\right\} \\
& C_{6}=\left\{s_{1} s_{2} s_{3}, s_{1} s_{3} s_{2}, s_{2} s_{1} s_{3}, s_{3} s_{2} s_{1}, s_{2} s_{3} s_{4}, s_{2} s_{4} s_{3}, s_{3} s_{2} s_{4}, s_{4} s_{3} s_{2}, s_{1} s_{2} s_{1} s_{4} s_{3},\right. \\
& s_{1} s_{2} s_{1} s_{3} s_{4}, s_{1} s_{2} s_{1} s_{3} s_{2}, s_{1} s_{2} s_{3} s_{2} s_{4}, s_{1} s_{2} s_{3} s_{4} s_{3}, s_{1} s_{2} s_{4} s_{3} s_{2}, s_{1} s_{3} s_{2} s_{1} s_{4}, \\
& s_{1} s_{3} s_{4} s_{3} s_{2}, s_{1} s_{4} s_{3} s_{2} s_{1}, s_{2} s_{1} s_{3} s_{2} s_{1}, s_{2} s_{1} s_{3} s_{4} s_{3}, s_{2} s_{3} s_{2} s_{1} s_{4}, s_{2} s_{3} s_{2} s_{4} s_{3}, \\
& s_{2} s_{4} s_{3} s_{2} s_{1}, s_{3} s_{2} s_{4} s_{3} s_{2}, s_{3} s_{4} s_{3} s_{2} s_{1}, s_{1} s_{2} s_{1} s_{3} s_{4} s_{3} s_{2}, s_{1} s_{2} s_{3} s_{2} s_{1} s_{4} s_{3}, \\
& \left.s_{1} s_{3} s_{2} s_{4} s_{3} s_{2} s_{1}, s_{2} s_{1} s_{3} s_{4} s_{3} s_{2} s_{1}, s_{1} s_{2} s_{1} s_{3} s_{2} s_{1} s_{4} s_{3} s_{2}, s_{2} s_{1} s_{3} s_{2} s_{1} s_{4} s_{3} s_{2} s_{1}\right\} \\
& C_{7}=\left\{s_{1} s_{2} s_{3} s_{4}, s_{2} s_{1} s_{3} s_{4}, s_{1} s_{3} s_{2} s_{4}, s_{1} s_{2} s_{4} s_{3}, s_{3} s_{2} s_{1} s_{4}, s_{2} s_{1} s_{4} s_{3}, s_{1} s_{4} s_{3} s_{2},\right. \\
& s_{4} s_{3} s_{2} s_{1}, s_{1} s_{2} s_{1} s_{3} s_{2} s_{4}, s_{1} s_{2} s_{3} s_{2} s_{4} s_{3}, s_{2} s_{3} s_{2} s_{1} s_{4} s_{3}, s_{3} s_{2} s_{1} s_{4} s_{3} s_{2}, \\
& s_{1} s_{2} s_{1} s_{4} s_{3} s_{2}, s_{2} s_{1} s_{3} s_{2} s_{1} s_{4}, s_{2} s_{1} s_{3} s_{2} s_{4} s_{3}, s_{1} s_{3} s_{2} s_{4} s_{3} s_{2}, s_{3} s_{2} s_{4} s_{3} s_{2} s_{1}, \\
& s_{2} s_{1} s_{4} s_{3} s_{2} s_{1}, s_{1} s_{2} s_{3} s_{2} s_{1} s_{4} s_{3} s_{2}, s_{2} s_{3} s_{2} s_{1} s_{4} s_{3} s_{2} s_{1}, s_{1} s_{3} s_{2} s_{1} s_{4} s_{3} s_{2} s_{1}, \\
& \left.s_{1} s_{2} s_{1} s_{3} s_{2} s_{1} s_{4} s_{3}, s_{1} s_{2} s_{1} s_{3} s_{2} s_{4} s_{3} s_{2}, s_{2} s_{1} s_{3} s_{2} s_{4} s_{3} s_{2} s_{1}\right\}
\end{aligned}
$$

As with types $A_{2}$ and $A_{3}$, we will write $\xi=\xi_{s}$ for all $s \in S$.
The first three conjugacy classes (in the above order) have the property that there are no same-length conjugations by simple reflections, which means that their corresponding class element in the Iwahori-Hecke algebra will simply be the
conjugacy class sum. The other class elements are all very large (as one would expect), so we will not diagramatically display them (although practically that is how they were calculated). However below we show the construction of the conjugacy class $C_{3}$, which is quite beautiful. Note that the method used to calculate the conjugacy class is the same as the method used to calculate the class element, but with $\xi=0$.

Will write $\Gamma_{i}$ for the class element corresponding to $C_{i}$, and $\tilde{T}_{i}$ for $\tilde{T}_{s_{i}}$ (writing 1 for the identity instead of $\left.\tilde{T}_{1}\right)$. The class elements of $\mathcal{H}\left(A_{4}\right)$ are then:

$$
\Gamma_{i}=\tilde{T}_{C_{i}} \quad \text { for } i=1,2,3
$$

$$
\Gamma_{4}=\tilde{T}_{C_{4}}+\xi\left(\tilde{T}_{121}+\tilde{T}_{232}+\tilde{T}_{343}+\tilde{T}_{12132}+\tilde{T}_{21321}+2 \tilde{T}_{12321}+2 \tilde{T}_{23432}+\tilde{T}_{23243}+\right.
$$

$$
\tilde{T}_{32432}+\tilde{T}_{1213432}+\tilde{T}_{2134321}+3 \tilde{T}_{1234321}+\tilde{T}_{1232143}+\tilde{T}_{1324321}+\tilde{T}_{121321432}+
$$

$$
\left.\tilde{T}_{213214321}\right)+\xi^{2}\left(\tilde{T}_{121321}+\tilde{T}_{232432}+\tilde{T}_{12134321}+\tilde{T}_{12324321}+\tilde{T}_{1213214321}\right)
$$

$$
\Gamma_{5}=\tilde{T}_{C_{5}}+\xi\left(\tilde{T}_{1214}+\tilde{T}_{1343}+\tilde{T}_{132143}+\tilde{T}_{213432}+\tilde{T}_{13214321}+2 \tilde{T}_{12134321}+\right.
$$

$$
2 \tilde{T}_{21321432}+\tilde{T}_{23214321}+\tilde{T}_{12132432}+\tilde{T}_{12321432}+\tilde{T}_{21324321}+\tilde{T}_{12324321}+
$$

$$
\left.\tilde{T}_{12132143}+2 \tilde{T}_{1213214321}\right)+\xi^{2}\left(\tilde{T}_{213214321}+\tilde{T}_{121321432}+\tilde{T}_{123214321}+\right.
$$

$$
\left.\tilde{T}_{121324321}\right)+\xi^{3} \tilde{T}_{1213214321}
$$

$$
\Gamma_{6}=\tilde{T}_{C_{6}}+\xi\left(\tilde{T}_{1213}+\tilde{T}_{1232}+\tilde{T}_{1321}+\tilde{T}_{2132}+\tilde{T}_{2321}+\tilde{T}_{2324}+\tilde{T}_{2343}+\tilde{T}_{2432}+\right.
$$

$$
\tilde{T}_{3243}+\tilde{T}_{3432}+2 \tilde{T}_{121321}+\tilde{T}_{121324}+2 \tilde{T}_{121343}+\tilde{T}_{121432}+2 \tilde{T}_{123214}+
$$

$$
\tilde{T}_{123243}+2 \tilde{T}_{123432}+2 \tilde{T}_{124321}+\tilde{T}_{132143}+\tilde{T}_{132432}+2 \tilde{T}_{134321}+\tilde{T}_{213214}+
$$

$$
\tilde{T}_{213243}+\tilde{T}_{213432}+\tilde{T}_{214321}+\tilde{T}_{232143}+2 \tilde{T}_{232432}+2 \tilde{T}_{234321}+\tilde{T}_{321432}+
$$

$$
\tilde{T}_{324321}+\tilde{T}_{12132143}+2 \tilde{T}_{12134321}+\tilde{T}_{12132432}+\tilde{T}_{12321432}+2 \tilde{T}_{12324321}+
$$

$$
\left.\tilde{T}_{13214321}+\tilde{T}_{21321432}+\tilde{T}_{21324321}+\tilde{T}_{23214321}+2 \tilde{T}_{1213214321}\right)+\xi^{2}\left(\tilde{T}_{12132}+\right.
$$

$$
\tilde{T}_{12321}+\tilde{T}_{21321}+\tilde{T}_{23243}+\tilde{T}_{23432}+\tilde{T}_{32432}+\tilde{T}_{1213214}+\tilde{T}_{1213243}+
$$

$$
2 \tilde{T}_{1213432}+\tilde{T}_{1214321}+2 \tilde{T}_{1232143}+\tilde{T}_{1232432}+3 \tilde{T}_{1234321}+\tilde{T}_{1321432}+
$$

$$
2 \tilde{T}_{1324321}+\tilde{T}_{2132143}+\tilde{T}_{2132432}+2 \tilde{T}_{2134321}+\tilde{T}_{2321432}+\tilde{T}_{2324321}+\tilde{T}_{3214321}+
$$

$$
\left.3 \tilde{T}_{121321432}+2 \tilde{T}_{121324321}+2 \tilde{T}_{123214321}+3 \tilde{T}_{213214321}\right)+
$$

$$
\xi^{3}\left(\tilde{T}_{121321}+\tilde{T}_{232432}+\tilde{T}_{12132143}+\tilde{T}_{12132432}+2 \tilde{T}_{12134321}+\tilde{T}_{12321432}+\right.
$$

$$
\begin{aligned}
& \left.2 \tilde{T}_{12324321}+\tilde{T}_{13214321}+\tilde{T}_{21321432}+\tilde{T}_{21324321}+\tilde{T}_{23214321}+4 \tilde{T}_{1213214321}\right)+ \\
& \xi^{4}\left(\tilde{T}_{121321432}+\tilde{T}_{121324321}+\tilde{T}_{123214321}+\tilde{T}_{213214321}\right)+\xi^{5} \tilde{T}_{1213214321} \\
& \Gamma_{7}=\tilde{T}_{C_{7}}+\xi\left(\tilde{T}_{12134}+\tilde{T}_{12143}+\tilde{T}_{12324}+\tilde{T}_{12343}+\tilde{T}_{12432}+\tilde{T}_{13214}+\tilde{T}_{13243}+\right. \\
& \quad \tilde{T}_{13432}+\tilde{T}_{14321}+\tilde{T}_{21324}+\tilde{T}_{21343}+\tilde{T}_{21432}+\tilde{T}_{23214}+\tilde{T}_{24321}+\tilde{T}_{32143}+ \\
& \tilde{T}_{34321}+2 \tilde{T}_{1213214}+2 \tilde{T}_{1213243}+\tilde{T}_{1213432}+2 \tilde{T}_{1214321}+\tilde{T}_{1232143}+ \\
& 2 \tilde{T}_{1232432}+2 \tilde{T}_{1321432}+\tilde{T}_{1324321}+2 \tilde{T}_{3214321}+2 \tilde{T}_{121321432}+3 \tilde{T}_{121324321}+ \\
& \left.3 \tilde{T}_{123214321}+2 \tilde{T}_{213214321}\right)+\xi^{2}\left(\tilde{T}_{121324}+\tilde{T}_{121343}+\tilde{T}_{121432}+\tilde{T}_{123214}+\right. \\
& \tilde{T}_{123243}+\tilde{T}_{123432}+\tilde{T}_{124321}+\tilde{T}_{132143}+\tilde{T}_{132432}+\tilde{T}_{134321}+\tilde{T}_{213214}+\tilde{T}_{213243}+ \\
& \tilde{T}_{213432}+\tilde{T}_{214321}+\tilde{T}_{232143}+\tilde{T}_{234321}+\tilde{T}_{321432}+\tilde{T}_{324321}+3 \tilde{T}_{12132143}+ \\
& 3 \tilde{T}_{12132432}+2 \tilde{T}_{12134321}+3 \tilde{T}_{12321432}+2 \tilde{T}_{12324321}+3 \tilde{T}_{13214321}+3 \tilde{T}_{21321432}+ \\
& \left.3 \tilde{T}_{21324321}+3 \tilde{T}_{23214321}+5 \tilde{T}_{1213214321}\right)+\xi^{3}\left(\tilde{T}_{1213214}+\tilde{T}_{1213243}+\tilde{T}_{1213432}+\right. \\
& \tilde{T}_{1214321}+\tilde{T}_{1232143}+\tilde{T}_{1232432}+\tilde{T}_{1234321}+\tilde{T}_{1321432}+\tilde{T}_{1324321}+\tilde{T}_{2132143}+ \\
& \tilde{T}_{2132432}+\tilde{T}_{2134321}+\tilde{T}_{2321432}+\tilde{T}_{2324321}+\tilde{T}_{3214321}+4 \tilde{T}_{121321432}+ \\
& \left.4 \tilde{T}_{121324321}+4 \tilde{T}_{123214321}+4 \tilde{T}_{213214321}\right)+\xi^{4}\left(\tilde{T}_{12132143}+\tilde{T}_{12132432}+\right. \\
& \tilde{T}_{12134321}+\tilde{T}_{12321432}+\tilde{T}_{12324321}+\tilde{T}_{13214321}+\tilde{T}_{21321432}+\tilde{T}_{21324321}+ \\
& \left.\tilde{T}_{23214321}+5 \tilde{T}_{1213214321}\right)+\xi^{5}\left(\tilde{T}_{121321432}+\tilde{T}_{121324321}+\tilde{T}_{123214321}+\right. \\
& \left.\tilde{T}_{213214321}\right)+\xi^{6} \tilde{T}_{1213214321} .
\end{aligned}
$$

Again, in the following diagram we will simplify notation by writing $i$ for $s_{i}$.


The conjugacy class $C_{3}$ in type $A_{4}$

### 5.5 Type $B_{3}$

Let $W$ be the Weyl group of type $B_{3}$, generated by $t, s_{1}, s_{2}$, and with relations $t^{2}=s_{i}^{2}=1,\left(t s_{1}\right)^{4}=\left(s_{1} s_{2}\right)^{3}=\left(t s_{2}\right)^{2}=1$. The conjugacy classes of $W$ are $C_{1}=\{1\}$,
$C_{2}=\left\{s_{2} t, s_{1} s_{2} t s_{1}, s_{1} s_{2} s_{1} t s_{1} s_{2}, t s_{1} s_{2} t s_{1} t, t s_{1} s_{2} s_{1} t s_{1} s_{2} t, s_{1} t s_{1} s_{2} t s_{1} t s_{1}\right\}$
$C_{3}=\left\{t, s_{1} t s_{1}, s_{2} s_{1} t s_{1} s_{2}\right\}, \quad C_{4}=\left\{s_{1}, s_{2}, s_{1} s_{2} s_{1}, t s_{1} t, t s_{1} s_{2} s_{1} t, s_{1} t s_{1} s_{2} s_{1} t s_{1}\right\}$,
$C_{5}=\left\{t s_{1} t s_{1}, s_{2} t s_{1} t s_{1} s_{2}, s_{1} s_{2} t s_{1} t s_{1} s_{2} s_{1}\right\}, \quad C_{6}=\left\{t s_{1} t s_{1} s_{2} s_{1} t s_{1} s_{2}\right\}$,
$C_{7}=\left\{s_{1} s_{2}, s_{2} s_{1}, t s_{2} s_{1} t, t s_{1} s_{2} t, s_{1} s_{2} s_{1} t s_{1} t, t s_{1} t s_{1} s_{2} s_{1}, t s_{2} s_{1} t s_{2} s_{1}, s_{1} s_{2} t s_{1} s_{2} t\right\}$,
$C_{8}=\left\{s_{1} t, t s_{1}, t s_{1} s_{2} s_{1}, s_{1} s_{2} s_{1} t, s_{1} t s_{1} s_{2}, s_{2} s_{1} t s_{1}\right\}$,
$C_{9}=\left\{s_{1} s_{2} t, s_{2} s_{1} t, t s_{1} s_{2}, t s_{2} s_{1}, s_{2} s_{1} t s_{2} s_{1}, s_{1} s_{2} t s_{1} s_{2}, t s_{1} t s_{2} s_{1} t s_{1}, s_{1} t s_{1} s_{2} t s_{1} t\right\}$,
$C_{10}=\left\{s_{2} s_{1} t s_{1} t, s_{1} s_{2} t s_{1} t, t s_{1} t s_{1} s_{2}, t s_{1} t s_{2} s_{1}, s_{1} s_{2} s_{1} t s_{2} t s_{1}, s_{2} t s_{1} t s_{1} s_{2} s_{1}\right\}$.
Again we abbreviate $\tilde{T}_{s_{i}}$ to $\tilde{T}_{i}$, and write $\tilde{T}_{C}$ for the conjugacy class sum. The
minimal basis for $Z\left(\mathcal{H}\left(B_{3}\right)\right)$ over $\mathbb{Z}[\xi]$ is the set $\left\{\Gamma_{1}, \ldots, \Gamma_{10}\right\}$, where the $\Gamma_{i}$ are:

$$
\begin{aligned}
\Gamma_{i}= & \tilde{T}_{C_{i}} \quad \text { for } i=1, \ldots, 6, \\
\Gamma_{7}= & \tilde{T}_{C_{7}}+\xi_{s}\left(\tilde{T}_{121}+\tilde{T}_{t 121 t}+\tilde{T}_{1 t 21 t 12}+\tilde{T}_{t 21 t 121}+2 \tilde{T}_{1 t 121 t 1}\right)+\xi_{t}\left(\tilde{T}_{t 1 t 121 t}+\tilde{T}_{t 1 t 21 t 1}\right)+ \\
& \xi_{s}^{2} \tilde{T}_{1 t 121 t 12}+\xi_{s} \xi_{t}\left(\tilde{T}_{t 1 t 21 t 12}+\tilde{T}_{t 1 t 121 t 1}\right), \\
\Gamma_{8}= & \tilde{T}_{C_{8}}+\xi_{s}\left(\tilde{T}_{121}+2 \tilde{T}_{21 t 12}+\tilde{T}_{121 t 1}+\tilde{T}_{1 t 121}+\tilde{T}_{t 121 t 1 t}+\tilde{T}_{t 1 t 121 t}\right)+\xi_{t}\left(\tilde{T}_{t 1 t}+\right. \\
& \left.\tilde{T}_{t 121 t}+\tilde{T}_{1 t 121 t 1}\right)+\xi_{s}^{2}\left(\tilde{T}_{121 t 12}+\tilde{T}_{t 1 t 21 t 12}+\tilde{T}_{t 1 t 121 t 1}\right), \\
\Gamma_{9}= & \tilde{T}_{C_{9}}+\xi_{s}\left(\tilde{T}_{21 t 1}+\tilde{T}_{121 t}+\tilde{T}_{12 t 1}+\tilde{T}_{t 121}+\tilde{T}_{1 t 12}+2 \tilde{T}_{121 t 12}+\tilde{T}_{t 1 t 21 t}+2 \tilde{T}_{t 1 t 21 t 12}+\right. \\
& \left.2 \tilde{T}_{t 1 t 121 t 1}\right)+\xi_{t}\left(\tilde{T}_{t 12 t}+\tilde{T}_{t 21 t}\right)+\xi_{s} \xi_{t}\left(\tilde{T}_{t 121 t}+2 \tilde{T}_{1 t 121 t 1}+\tilde{T}_{t 121 t 12}+\tilde{T}_{21 t 121 t}\right)+ \\
& \xi_{s}^{2}\left(\tilde{T}_{121 t 1}+\tilde{T}_{1 t 121}+\tilde{T}_{21 t 12}+\tilde{T}_{t 121 t 1 t}+\tilde{T}_{t 1 t 121 t}\right)+\xi_{s}^{3}\left(\tilde{T}_{121 t 12}+\tilde{T}_{t 1 t 21 t 12}+\right. \\
& \left.\tilde{T}_{t 1 t 121 t 1}\right)+\xi_{s}^{2} \xi_{t} \tilde{T}_{1 t 21 t 121}+\xi_{s} \xi_{t}^{2}\left(\tilde{T}_{t 1 t 21 t 12}+\tilde{T}_{t 1 t 121 t 1}\right), \\
\Gamma_{10}= & \tilde{T}_{C_{10}}+\xi_{s}\left(\tilde{T}_{121 t 1 t}+\tilde{T}_{t 121 t 1}+\tilde{T}_{t 21 t 12}+\tilde{T}_{1 t 121 t}+\tilde{T}_{t 1 t 121}+2 \tilde{T}_{1 t 21 t 121}\right)+\xi_{t}\left(\tilde{T}_{t 1 t 21 t}+\right. \\
& \left.\tilde{T}_{t 1 t 121 t 1}+\tilde{T}_{t 1 t 21 t 12}\right)+\xi_{s} \xi_{t}\left(\tilde{T}_{t 1 t 21 t 1}+\tilde{T}_{t 1 t 121 t}\right)+\xi_{s}^{2}\left(\tilde{T}_{1 t 21 t 12}+\tilde{T}_{1 t 121 t 1}+\right. \\
& \left.\tilde{T}_{t 121 t 12}\right)+\xi_{s}^{2} \xi_{t}\left(\tilde{T}_{t 1 t 121 t 1}+\tilde{T}_{t 1 t 21 t 12}\right)+\xi_{s}^{3} \tilde{T}_{1 t 21 t 121} .
\end{aligned}
$$

We show below the construction of $\Gamma_{7}$ in type $B_{3}$, where we abbreviate $\tilde{T}_{s_{i}}$ simply to $i$.


## Chapter 6 The Brauer homomorphism

### 6.1 Further properties of the class elements

For our results on the Brauer homomorphism, we need to derive some more results about the class elements and the minimal basis. In particular, as a consequence of the algorithm, one has additional information about the terms $r_{w} \tilde{T}_{w}$ which appear in the class element $\Gamma_{C}$.
(6.1.1) Lemma. Let $d$ be distinguished in $\langle s\rangle d\langle s\rangle$, for $s \in S$, such that $d s \neq s d$, and let $s_{i} \in S$. Then

$$
\begin{aligned}
& s_{i} \leq d \Longrightarrow s_{i} \leq s d s, \\
& s_{i} \leq d s \Longrightarrow s_{i} \leq s d \text { and } s_{i} \leq s d s .
\end{aligned}
$$

where the inequality is the Bruhat order.
Proof. We may assume $s_{i}$ is not equal to $s$, as otherwise the lemma follows trivially. Since $s d \neq d s$, we have $l(s d s)=l(d)+2$, so the first implication follows. On the other hand, if $s_{i} \leq s d$, then $s_{i}$ must be less than $d$ in the Bruhat order, and so less than $s d$ and $s d s$, since $l(s d)=l(d)+1$.

The contrapositive of the above is actually very useful, so we state it seperately.
(6.1.2) Lemma. Let $d$ be distinguished in $\langle s\rangle d\langle s\rangle$, for $s \in S$, such that $d s \neq s d$, and let $s_{i} \in S$. Then

$$
\begin{gathered}
s_{i} \not \leq s d s \Longrightarrow s_{i} \not \leq d, d s, \text { or } s d, \\
s_{i} \not \leq s d \Longrightarrow s_{i} \not \leq d s \text { or } d .
\end{gathered}
$$

If a term $r \tilde{T}_{w}$ appears with $r \neq 0$ in a class element $\Gamma_{C}$, it is probably clear what is meant when we say " $\tilde{T}_{w}$ arises from $\tilde{T}_{C_{\text {min }}}$ " in the context of additions via the algorithm $\mathfrak{A}$. We want to define this more explicitly.
(6.1.3) Definition. Let $w \in W$, and $u \in C_{\min }$. Then $\tilde{T}_{w}$ is said to arise from $\tilde{T}_{u}$ if there exists a sequence $w_{0}, w_{1}, \ldots, w_{m}$ of elements of $W$ with $u=w_{0}$ and
$w=w_{m}$, and a sequence of simple reflections $s_{0}, \ldots, s_{m-1}$ from $S$ such that for each $i$ and $d_{i}$ distinguished in $\left\langle s_{i}\right\rangle d_{i}\left\langle s_{i}\right\rangle$, we have $l\left(s_{i} d_{i} s_{i}\right)=l\left(d_{i}\right)+2$ and either:
(1) $w_{i}=d_{i}$ and $w_{i+1}=s_{i} d_{i} s_{i}$, or
(2) $w_{i}=d_{i} s_{i}$ and $w_{i+1}=s_{i} d_{i}$ or $s_{i} d_{i} s_{i}$, or
(3) $w_{i}=s_{i} d_{i}$ and $w_{i+1}=d_{i} s_{i}$ or $s_{i} d_{i} s_{i}$.
(6.1.4) Lemma. Suppose $\tilde{T}_{w}$ arises from $\tilde{T}_{u}$ with $u \in C_{\min }$, and let $w_{0}, \ldots, w_{m}$ be the sequence linking them, with $s_{0}, \ldots, s_{m-1}$ the corresponding sequence of simple reflections. Then $s_{j} \leq w_{i+1}$ for all $0 \leq j<i \leq m-1$.

Proof. By induction on either $m$ or $i$, and using (6.1.1).
Again, we will need a contrapositive:
(6.1.5) Corollary. Define $u, w$, and the sequences of $w_{i}$ and $s_{i}$ as in (6.1.4). Let $s_{l} \in S$, with $s_{l} \neq s_{i}$ for any $0 \leq i \leq m-1$. Then $s_{l} \not \leq w$ implies $s_{l} \not \leq u$.
(6.1.6) Proposition. Let $C$ be a conjugacy class in $W, \Gamma_{C}$ be the corresponding class element, and $s \in S$. Then $s \leq w$ for all $w \in C_{\min }$ implies $s \leq w$ for all $r \tilde{T}_{w} \leq \Gamma_{C}$. Similarly, $s \not \leq w$ for some $r \tilde{T}_{w} \leq \Gamma_{C}$ implies $s \not \leq u$ for some $\tilde{T}_{u} \leq \tilde{T}_{C_{\text {min }}}$.

Proof. It is clear from the definition that $\tilde{T}_{w}$ appears in $\Gamma_{C}$ with non-zero coefficient if and only if it arises from $\tilde{T}_{u}$ for some $u \in C_{\text {min }}$. The key observation here is that in the sequence of $w_{i}$ described above, if a simple reflection $s$ is less than $w_{i}$ then it is also less than $w_{i+1}$, by lemma (6.1.1). And so, if $s \leq u \in C_{\min }$, then $s$ is less than every term which arises from $\tilde{T}_{u}$. On the other hand, if $s$ is not less than a term $r \tilde{T}_{w}$ in $\Gamma_{C}$, then there must be a term in $\tilde{T}_{C_{\text {min }}}$ from which it arose, which also does not contain $s$.

### 6.2. The Jones Brauer homomorphism.

In his thesis [J1], Lenny Jones defined a Brauer type homomorphism from the centre of the Iwahori-Hecke algebra of type $A_{n-1}$ into the centre of a certain parabolic subalgebra. This Brauer homomorphism has already found applications in the Green correspondence for $\mathcal{H}$-modules in [D], and has been extended to a

Braur homomorphism for $q$-Schur algebras in [DD]. In this chapter, we will describe the image of the minimal basis under the Brauer homomorphism, and the kernel of the homomorphism in terms of the minimal basis.

We begin with the definition as given in [J1].
Let $\mathcal{H}$ be the Iwahori-Hecke algebra of type $A_{n-1}$, and let $\lambda$ be a partition of $n$. Then we write $\mathcal{H}_{\lambda}$ for the subalgebra of $\mathcal{H}$ corresponding to the subalgebra $W_{\lambda}$ of $W=W\left(A_{n}\right)$. We will be dealing with a few special examples, so we define the following abbreviations. Let $\gamma=(l, n-l) \vdash n$, and write $W_{l}=W_{\left(l, 1^{n-l}\right)}$ and $\mathcal{H}_{l}=\mathcal{H}_{\left(l, 1^{n-l}\right)}$. For clarity, we will continue to write $W_{\left(1^{l}, n-l\right)}$ and $\mathcal{H}_{\left(1^{l}, n-l\right)}$ in full.

The Brauer homomorphism is a composition of two maps. Firstly, we may decompose the centralizer of the subalgebra as follows:

$$
Z_{\mathcal{H}}\left(\mathcal{H}_{\gamma}\right)=Z\left(\mathcal{H}_{\gamma}\right) \oplus \bigoplus_{x \notin W_{\gamma}}\left(Z_{\mathcal{H}_{l} \tilde{T}_{x} \mathcal{H}_{l}}\left(\mathcal{H}_{\gamma}\right)\right)
$$

Since the centre is a subalgebra of the centralizer $Z_{\mathcal{H}}\left(\mathcal{H}_{\gamma}\right)$, we may project $Z(\mathcal{H})$ onto $Z\left(\mathcal{H}_{\gamma}\right)$. Let us call this projection $\rho$.

The Brauer homomorphism $\sigma$ is defined to be the composition of $\rho$ with the canonical homomorphism $\theta: Z\left(\mathcal{H}_{\gamma}\right) \rightarrow Z\left(\mathcal{H}_{\gamma}\right) /\left[N_{W_{l}, 1}\left(\mathcal{H}_{\gamma}\right) \cap Z\left(\mathcal{H}_{\gamma}\right)\right]$, giving us a homomorphism:

$$
\sigma: Z(\mathcal{H}) \longrightarrow Z\left(\mathcal{H}_{\gamma}\right) /\left[N_{W_{l}, 1}\left(\mathcal{H}_{\gamma}\right) \cap Z\left(\mathcal{H}_{\gamma}\right)\right] .
$$

### 6.3. The image of the minimal basis

Earlier in this thesis we have discussed centralizers of parabolic subalgebras of the Iwahori-Hecke algebra. But the Brauer homomorphism maps not to a centralizer but the centre of a parabolic subalgebra. In this case, all the results of chapter three relating to the algorithm and the minimal basis are correct - a fact easily seen as follows.
(6.3.1) Lemma. Let $C$ be a conjugacy class of $W_{\gamma}$ where $\gamma=(l, n-l)$. Then $C=C_{1} C_{2}$ where $C_{1}$ is a conjugacy class in $W_{l}$ and $C_{2}$ is a conjugacy class in $W_{\left(1^{l}, n-l\right)}$. Similarly, if $\Gamma_{C}$ is a class element in $\mathcal{H}_{\gamma}$, then $\Gamma_{C}=\Gamma_{C_{1}} \Gamma_{C_{2}}$.

Proof. This follows by noting that $W_{l}$ and $W_{\left(1^{l}, n-l\right)}$ commute, so conjugation in $W_{\gamma}$ is actually two separate commuting conjugations, of the two separate components of any element of $W_{\gamma}$. The $\xi$-analogue follows also.

There are some useful characteristics of the minimal basis which help us describe its image under $\sigma$. One of these in particular is that expressed in lemma (6.1.6), that if a generator ( $s_{l}$ for example) does not appear in a term in a class element, then the term has arisen via additions in the algorithm from a minimal element also without the generator $s_{l}$. This gives us:
(6.3.2) Lemma. The image of $\Gamma_{C}$ under the projection $\rho$ is a sum of class elements of $\mathcal{H}_{\gamma}$, each with coefficient one.

Proof. The image $\rho\left(\Gamma_{C}\right)$ must be an element of $Z\left(\mathcal{H}_{\gamma}\right)$, and so a linear combination of class elements of $\mathcal{H}_{\gamma}$. Now suppose one of these class elements of $\mathcal{H}_{\gamma}$ has coefficient not equal to one, and let $C_{\gamma}$ be the corresponding conjugacy class in $W_{\gamma}$. It suffices to show that $C_{\gamma, \text { min }} \subseteq C_{\text {min }}$.

Let $u \in C_{\gamma, \min }$, not in $C_{\min }$. Then $u$ has arisen in $\Gamma_{C}$ from a shortest element of $C$ via a sequence of reflections $s_{0}, \ldots, s_{m}$, which must include $s_{l}$ - otherwise, the element of $C_{\min }$ giving rise to $u$ would also have to be in $W_{\gamma}$ by (6.1.5), and $u$ could not be minimal in $C_{\gamma}$. But then, we must have $s_{l} \leq u$ by (6.1.4), a contradiction.

If $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots\right) \vdash n$, then we call the elements $\lambda_{i}$ the components of $\lambda$. An $(l, n-l)$-bipartition is a pair of partitions $(\mu, \nu)$, with $\mu \vdash l$ and $\nu \vdash n-l$. An ( $l, n-l$ )-bipartition of $\lambda$ is an $(l, n-l)$-bipartition whose components are those of $\lambda$. Let $\operatorname{Bip}(\lambda)=\left\{\left(\mu_{i}, \nu_{i}\right)\right\}$ be the set of $(l, n-l)$-bipartitions of $\lambda \vdash n$.

An element $w \in C_{\lambda}$ is said to be of shape $\lambda=\left(\lambda_{1}, \ldots, \lambda_{r}\right)$ if it is in $C_{\lambda, \min }$ and also in $W_{\left(\lambda_{1}, 1^{n-\lambda_{1}}\right)} \times W_{\left(1_{1}^{\lambda}, \lambda_{2}, 1^{\left.n-\lambda_{1}-\lambda_{2}\right)}\right.} \times \cdots \times W_{\left.1^{n-\lambda_{r}}, \lambda_{r}\right)}$.
(6.3.3) Proposition. The number of conjugacy classes of $W_{\gamma}$ contained in $C_{\lambda} \in$ $\operatorname{ccl}(W)(f o r \lambda \vdash n)$ is the number of $(l, n-l)$-bipartitions of $\lambda$. This is also thus the number of class elements of $\mathcal{H}_{\gamma}$ contained in the class element $\Gamma_{C_{\lambda}}$ of $\mathcal{H}$.

Proof. We show there is a one-to-one correspondence between ( $l, n-l$ )-bipartitions
of $\lambda$, and conjugacy classes of $W_{\gamma}$ contained in $C_{\lambda}$.
Let $(\mu, \nu)$ be an $(l, n-l)$-bipartition of $\lambda$. Then there is a composition $\mu \nu$ of $n$ with the components of $\mu$ followed by those of $\nu$ which is conjugate to the partition $\lambda$. Thus the element of shape $\mu \nu$ is conjugate to the element of shape $\lambda$, and is an element of $C_{\lambda, \min }$. It also corresponds uniquely to a conjugacy class of $W_{\gamma}$ which is contained in $C_{\lambda}$.

Any conjugacy class of $W_{\gamma}$ contained in $C_{\lambda}$ can be written as a product of a conjugacy class of $W_{l}$ and one of $W_{\left(1^{l}, n-l\right)}$. These in turn correspond uniquely to partitions $\mu$ and $\nu$ of $l$ and $n-l$ respectively, and there are shortest elements from each of these conjugacy classes of shape $\mu$ and $\nu$, whose product is a shortest element of $C_{\lambda}$ of shape $\mu \nu$. Then $(\mu, \nu)$ must be an $(l, n-l)$-bipartition of $\lambda$, since each shortest element of the conjugacy class $C_{\lambda}$ corresponds to a composition of $\lambda$ - meaning the components of $(\mu, \nu)$ are a composition of $\lambda$.
(6.3.4) Corollary. Let $\lambda \vdash n$. Then

$$
\rho\left(\Gamma_{\lambda}\right)=\sum_{\left(\mu_{i}, \nu_{i}\right) \in \operatorname{Bip}(\lambda)} \Gamma_{C_{\mu_{i}}} \Gamma_{C_{\nu_{i}}}
$$

where $C_{\mu_{i}}$ and $C_{\nu_{i}}$ are conjugacy classes in $W_{l}$ and $W_{\left(1^{l}, n-l\right)}$ respectively.
Proof. This follows from (6.3.2) and (6.3.3).
(6.3.5) Lemma. Let $w \in C_{\min }$ for some conjugacy class $C$ of $W_{l}$. Then $\Gamma_{C}<$ $N_{W_{l}, 1}\left(\tilde{T}_{w}\right)$.

Proof. Firstly, we know $\Gamma_{C} \leq N_{W_{l}, 1}\left(\tilde{T}_{w}\right)$, since $N_{W_{l}, 1}\left(\tilde{T}_{w}\right)$ is both central in $\mathcal{H}_{l}$ and contains shortest elements of $C$, so satisfies the requirements of (3.4.1). So it suffices to show that either the shortest elements of other conjugacy classes also occur in $N_{W_{l}, 1}\left(\tilde{T}_{w}\right)$, or the coefficient of $\tilde{T}_{C}$ in $N_{W_{l}, 1}\left(\tilde{T}_{w}\right)$ is more than one.

Suppose that $w$ is a Coxeter element of $W_{l}$, and $C$ is the Coxeter class. Then the element $N_{W_{l-1}, 1}\left(\tilde{T}_{w}\right)$ satisfies (3.2.1.1) and (3.2.1.2), by [J2], so is in fact $\Gamma_{C}$ by (3.2.6). Then

$$
\begin{aligned}
N_{W_{l}, 1}\left(\tilde{T}_{w}\right) & =N_{W_{l}, W_{l-1}}\left(N_{W_{l-1}, 1}\left(\tilde{T}_{w}\right)\right) \\
& =N_{W_{l}, W_{l-1}}\left(\Gamma_{C}\right) \\
& >\Gamma_{C}
\end{aligned}
$$

where the first equality is by $[\mathrm{J} 2 ; 2.12]$.
If $w$ is not a Coxeter element of $W_{l}$ and $C$ is not the Coxeter class, then $w$ is a Coxeter element of some parabolic subgroup $W_{\alpha}$ of $W_{l}$ (where $\alpha$ is a partition of $l)$. We write $\alpha=\left(\alpha_{1}, \ldots, \alpha_{t}\right)$. The Jones basis element of the centre of $\mathcal{H}_{l}$ corresponding to this conjugacy class of $W_{l}$ is

$$
b_{\alpha}=N_{W_{l}, W_{\alpha}}\left(\eta_{\alpha}\right)
$$

where

$$
\eta_{\alpha}=\prod_{i=1}^{t} N_{W_{\left(1, \ldots, 1, \alpha_{i}-1,1, \ldots, 1\right)}, 1}\left(\tilde{T}_{w_{i}}\right)
$$

and where $w_{i}$ is the Coxeter element of the component $W_{\alpha_{i}}$ of the parabolic subgroup $W_{\alpha}=W_{\alpha_{1}} \times \cdots \times W_{\alpha_{t}}$.

But then $\eta_{\alpha} \leq N_{W_{\alpha}, 1}\left(\tilde{T}_{w}\right)$, so

$$
\begin{aligned}
N_{W_{l}, 1}\left(\tilde{T}_{w}\right) & =N_{W_{l}, W_{\alpha}}\left(N_{W_{\alpha}, 1}\left(\tilde{T}_{w}\right)\right) \\
& \geq N_{W_{l}, W_{\alpha}}\left(\eta_{\alpha}\right) \\
& =b_{\alpha} .
\end{aligned}
$$

By $[\mathrm{J} 2 ; 3.29],\left.b_{\alpha}\right|_{\xi=0}=\left[N_{W_{l}}\left(W_{\alpha}\right): W_{\alpha}\right] \tilde{T}_{C}$ - that is, the integer part of the coefficient of $\tilde{T}_{C}$ in $b_{\alpha}$ is $\left[N_{W_{l}}\left(W_{\alpha}\right): W_{\alpha}\right]$ (Jones further shows that this is the only coefficient of $\tilde{T}_{C}$ in $b_{\alpha}$ in (3.28) and (3.30), but we do not need this). Since we have assumed $C=C_{\alpha}$ is not the Coxeter class, we have that $\left[N_{W_{l}}\left(W_{\alpha}\right): W_{\alpha}\right]>1$, and so the coefficient of $\tilde{T}_{C}$ in $N_{W_{l}, 1}\left(\tilde{T}_{w}\right)$ is more than one, and so we must have $N_{W_{l}, 1}\left(\tilde{T}_{w}\right)>\Gamma_{C}$.

Now, we only need to conclude that nothing is lost from any one $\Gamma_{C}$ under $\theta$.
(6.3.6) Corollary. There are no elements of $N_{W_{l}, 1}\left(\mathcal{H}_{\gamma}\right)$ less than any class element $\Gamma_{C}$.

Therefore,

$$
\sigma\left(\Gamma_{C}\right)=\rho\left(\Gamma_{C}\right) \bmod N_{W_{l}, 1}\left(\mathcal{H}_{\gamma}\right) \cap Z\left(\mathcal{H}_{\gamma}\right) .
$$

(6.3.7) Example. Let $n=4$, and $l=2$. Then we are dealing with the Weyl group of type $A_{3}$, and the subgroup $W_{(2,2)}=W\left(A_{1}\right) \times W\left(A_{1}\right)=\left\langle s_{1}, s_{3}\right\rangle$. The
class elements are described in section 5.2, and using the notation from there, we have

$$
\begin{aligned}
Z(\mathcal{H}) & \stackrel{\rho}{\longrightarrow}\left(\mathcal{H}_{\left\langle s_{1}, s_{3}\right\rangle}\right) \\
\Gamma_{C_{i d}} & \longrightarrow \tilde{T}_{i d} \\
\Gamma_{C_{1}} & \longrightarrow \tilde{T}_{s_{1}}+\tilde{T}_{s_{3}} \\
\Gamma_{C_{13}} & \longrightarrow \tilde{T}_{s_{1} s_{3}} \\
\Gamma_{C_{12}} & \longrightarrow 0 \\
\Gamma_{C_{123}} & \longrightarrow 0
\end{aligned}
$$

Now $N_{W_{2}, 1}\left(\mathcal{H}_{\gamma}\right)$ is spanned by $N_{W_{2}, 1}\left(\tilde{T}_{w}\right)$ for $w \in W_{(2,2)}=\left\langle s_{1}, s_{3}\right\rangle$, and these elements can be simply calculated (in such a small case):

$$
\begin{aligned}
N_{W_{2}, 1}\left(\tilde{T}_{i d}\right) & =\tilde{T}_{i d} \tilde{T}_{i d} \tilde{T}_{i d}+\tilde{T}_{s_{1}} \tilde{T}_{i d} \tilde{T}_{s_{1}} \\
& =\tilde{T}_{i d}+\tilde{T}_{s_{1}} \tilde{T}_{s_{1}} \\
& =2 \tilde{T}_{i d}+\xi \tilde{T}_{s_{1}} \\
N_{W_{2}, 1}\left(\tilde{T}_{s_{1}}\right) & =\tilde{T}_{i d} \tilde{T}_{s_{1}} \tilde{T}_{i d}+\tilde{T}_{s_{1}} \tilde{T}_{s_{1}} \tilde{T}_{s_{1}} \\
& =\tilde{T}_{s_{1}}+\tilde{T}_{s_{1}}\left(\tilde{T}_{i d}+\xi \tilde{T}_{s_{1}}\right) \\
& =\xi \tilde{T}_{i d}+\left(2+\xi^{2}\right) \tilde{T}_{s_{1}} \\
N_{W_{2}, 1}\left(\tilde{T}_{s_{3}}\right) & =N_{W_{2}, 1}\left(\tilde{T}_{i d}\right) \tilde{T}_{s_{3}} \\
& =2 \tilde{T}_{s_{3}}+\xi \tilde{T}_{s_{1} s_{3}} \\
N_{W_{2}, 1}\left(\tilde{T}_{s_{1} s_{3}}\right) & =N_{W_{2}, 1}\left(\tilde{T}_{s_{1}}\right) \tilde{T}_{s_{3}} \\
& =\xi \tilde{T}_{s_{3}}+\left(2+\xi^{2}\right) \tilde{T}_{s_{1} s_{3}} .
\end{aligned}
$$

Note that none if these is less than $\rho\left(\Gamma_{C}\right)$ for any $C$, as in (6.3.6).
We can check that $\sigma$ is a homomorphism in a few interesting cases now. For instance,

$$
\begin{aligned}
\rho\left(\Gamma_{C_{1}} \Gamma_{C_{13}}\right) & =\tilde{T}_{1}+\tilde{T}_{3}+2 \xi \tilde{T}_{s_{1} s_{3}} \\
& =\left(\tilde{T}_{s_{1}}+\tilde{T}_{s_{3}}\right) \tilde{T}_{s_{1} s_{3}} \\
& =\rho\left(\Gamma_{C_{1}}\right) \rho\left(\Gamma_{C_{13}}\right) .
\end{aligned}
$$

A more challenging example is with $\Gamma_{C_{1}}$ and $\Gamma_{C_{12}}$ :

$$
\begin{aligned}
\sigma\left(\Gamma_{C_{1}} \Gamma_{C_{12}}\right)= & \theta\left(4 \xi \tilde{T}_{i d}+\left(4+3 \xi^{2}\right) \tilde{T}_{s_{1}}+\left(4+3 \xi^{2}\right) \tilde{T}_{s_{3}}+\left(4 \xi+2 \xi^{3}\right) \tilde{T}_{s_{1} s_{3}}\right) \\
= & \theta\left(\xi\left[2 \tilde{T}_{i d}+\xi \tilde{T}_{s_{1}}\right]+2\left[\xi \tilde{T}_{i d}+\left(2+\xi^{2}\right) \tilde{T}_{s_{1}}\right]\right. \\
& \left.+\left(2+\xi^{2}\right)\left[2 \tilde{T}_{s_{3}}+\xi \tilde{T}_{s_{1} s_{3}}\right]+\xi\left[\xi \tilde{T}_{s_{3}}+\left(2+\xi^{2}\right) \tilde{T}_{s_{1} s_{3}}\right]\right) \\
= & 0,
\end{aligned}
$$

which is as it should be, since $\sigma\left(\Gamma_{C_{1}}\right) \sigma\left(\Gamma_{C_{12}}\right)=\left(\tilde{T}_{s_{1}}+\tilde{T}_{s_{3}}\right) 0=0$.

### 6.4. The kernel of the Brauer homomorphism

(6.4.1) Lemma. $N_{W_{l}, 1}\left(h \Gamma_{C}\right)=N_{W_{l}, 1}(h) \Gamma_{C} \in Z\left(\mathcal{H}_{\gamma}\right)$ for $C$ a conjugacy class in $W_{\left(1^{l}, n-l\right)}$, and $h \in \mathcal{H}_{l}$. Thus $N_{W_{l}, 1}(h) \Gamma_{C} \in \operatorname{ker} \theta$ for all $h \in \mathcal{H}_{l}$ and $C \in$ $\operatorname{ccl}\left(W_{\left(1^{l}, n-l\right)}\right)$.

Proof. This follows since the conjugations by elements of $\mathcal{H}_{l}$ in the definition of $N_{W_{l}, 1}$ all commute with $\Gamma_{C} \in \mathcal{H}_{\left(1^{l}, n-l\right)}$. Then since $N_{W_{l}, 1}(h) \in Z\left(\mathcal{H}_{l}\right)$, we have $N_{W_{l}, 1}\left(h \Gamma_{C}\right)=N_{W_{l}, 1}(h) \Gamma_{C} \in Z\left(\mathcal{H}_{\gamma}\right)$, and so is in the kernel of $\theta$ for any $h \in \mathcal{H}_{l}$.

We may write $N_{W_{l}, 1}\left(\tilde{T}_{w}\right)$ for $w \in W_{l}$ as a linear combination of the class elements in $\mathcal{H}_{l}$, with coefficients in $R^{+}$(since the class elements are minimal), as follows:

$$
N_{W_{l}, 1}\left(\tilde{T}_{w}\right)=\sum_{\mu_{i} \vdash l} a_{w, i} \Gamma_{C_{\mu_{i}}}
$$

with $a_{w, i} \in R, C_{\mu_{i}}$ conjugacy classes in $W_{l}$, and $\Gamma_{C_{\mu_{i}}}$ the corresponding class elements in $\mathcal{H}_{l}$.

Let $C_{\nu}$ be a conjugacy class in $W_{\left(1^{l}, n-l\right)}$, with $\nu \vdash n-l$. Then for each $\mu_{i} \vdash l$ there exists a unique partition $\lambda_{i}$ of $n$ with components from $\mu_{i}$ and $\nu$. Correspondingly, there is a unique conjugacy class in $W$ for each pair $\mu_{i}$ and $\nu$. This class we denote $C_{\mu_{i}, \nu}$ or $C_{\lambda_{i}}$.

Then for a fixed $\nu \vdash n-l$ and with the $a_{w, i}$ as in $\star$,

$$
\begin{aligned}
\sigma\left(\sum_{\mu_{i} \vdash l} a_{w, i} \Gamma_{C_{\lambda_{i}}}\right) & =\theta\left(\sum_{\mu_{i} \vdash l} a_{w, i} \Gamma_{C_{\mu_{i}}} \Gamma_{C_{\nu}}\right) \\
& =\theta\left(\left(\sum_{\mu_{i} \vdash l} a_{w, i} \Gamma_{C_{\mu_{i}}}\right) \Gamma_{C_{\nu}}\right) \\
& =\theta\left(N_{W_{l}, 1}\left(\tilde{T}_{w}\right) \Gamma_{C_{\nu}}\right) \\
& =\theta\left(N_{W_{l}, 1}\left(\tilde{T}_{w} \Gamma_{C_{\nu}}\right)\right) \\
& \in \theta\left(N_{W_{l}, 1}\left(\mathcal{H}_{\gamma}\right) \cap Z\left(\mathcal{H}_{\gamma}\right)\right) \\
& =0 .
\end{aligned}
$$

The kernel of the map $\sigma$ is the set of those central elements which either project to zero under the map $\rho$, or whose projection under $\rho$ is in $N_{W_{l}, 1}\left(\mathcal{H}_{\gamma}\right)$ (the projection of a central element will automatically also be in $\left.Z\left(\mathcal{H}_{\gamma}\right)\right)$. Those that project under $\rho$ to zero are relatively easy to describe with reference to the class elements in $Z(\mathcal{H})$, as follows. Let $\Gamma_{C_{\lambda}}$ be a class element, with $C_{\lambda}$ a conjugacy class of $W$. Then $\rho\left(\Gamma_{C_{\lambda}}\right)=0$ if and only if $C \cap W_{l} \times W_{\left(1^{l}, n-l\right)}=\emptyset$, which happens if and only if there are no $\gamma$-bipartitions of $\lambda$.

If $\rho(h) \neq 0$ for $h \in Z(\mathcal{H})$, then $\sigma(h)=0$ if and only if $\rho(h) \in N_{W_{l}, 1}\left(\mathcal{H}_{\gamma}\right)=$ $N_{W_{l}, 1}\left(\mathcal{H}_{l}\right) \times Z\left(\mathcal{H}_{\left(1^{l}, n-l\right)}\right)$. This means that $\rho(h)$ must be in the span of terms of form $\sum_{\mu_{i} \vdash l} a_{w, i} \Gamma_{\mu_{i}} \Gamma_{\nu}$, where $\nu \vdash n-l$, and $C_{\nu}$ is a conjugacy class in $W_{\left(1^{l}, n-l\right)}$, since $h$ is in the $R$-span of the set of class elements. This can only happen if $h$ is a linear combination of terms of form $\sum_{\mu_{i} \vdash l} a_{w, i} \Gamma_{C_{\lambda_{i}}}$ where the $\lambda_{i}$ are the partitions of $n$ corresponding to the compositions ( $\mu_{i}, \nu$ ) of $n$.

Thus we have shown the following.
(6.4.2) Theorem. Let the set $\left\{\mu_{i} \mid 1 \leq i \leq r\right\}$ be the complete set of partitions of $l$, and for each $\nu \vdash n-l$ let $C_{\mu_{i}, \nu}$ be the conjugacy class in $W$ corresponding to the composition of $n$ with components from $\mu_{i}$ and $\nu$.

Then the kernel of the Brauer homomorphism $\sigma$ is the set of elements spanned by those of form

$$
\sum_{i=1}^{r} a_{w, i} \Gamma_{C_{\mu_{i}, \nu}}
$$

for all $w \in W_{l}$, and for all $\nu \vdash n-l$, together with all those $\Gamma_{C_{\lambda}}$ where $\lambda$ has no ( $l, n-l)$-bipartitions.

### 6.5. A Conjecture on the minimal basis

We have now given a generalization of (2.2.1) in the following cases:
(1) $W$ is of type $H_{3}$ or is dihedral, and $J=S$;
(2) $W$ is any Coxeter group and $|J|=1$;
(3) $W$ is a Weyl group and $|J|=2$;
(4) $W$ is of type $A$ or $B$ and $J$ is principal.

This is sufficient motivation to make the following general conjecture:
(6.5.1) Conjecture. Let $W$ be any Coxeter group generated by the set $S$ of simple reflections, and let $J \subseteq S$. Suppose conjecture (2.2.1) holds. Then
(i) $Z_{\mathcal{H}}\left(\mathcal{H}_{J}\right)_{\min }^{+}$is an $R$-basis for $Z_{\mathcal{H}}\left(\mathcal{H}_{J}\right)$,
(ii) $h \in Z_{\mathcal{H}}\left(\mathcal{H}_{J}\right)_{\min }^{+}$if and only if
a) $\left.h\right|_{\xi_{s}=0}=\tilde{T}_{\mathbb{C}}$ for some $\mathfrak{C} \in c c l_{J}(W)$, and
b) $h-\tilde{T}_{\mathbb{C}}$ contains no shortest elements of any $J$-conjugacy class.

Further, by the remarks in 3.5 and the results in chapter 3, the proof of this conjecture is reduced to proving the following conjecture, an analogy of (1.1.2):

## (6.5.2) Conjecture.

(i) If $J \subseteq S$, then every $W_{J}-W_{J}$ double coset is reducible.
(ii) If $\mathfrak{C}$ is a J-conjugacy class, and $w, w^{\prime} \in \mathfrak{C}_{\min }$, then there exists a sequence of $x_{i} \in W_{J}$ and $w_{i} \in \mathfrak{C}_{\min }$ such that $w=w_{0}$ and $w_{n}=w^{\prime}$, with $x_{i} w_{i} x_{i}^{-1}=w_{i+1}$ and $l\left(x_{i} w_{i}\right)=l\left(x_{i}\right)+l\left(w_{i}\right)$ or $l\left(w_{i} x_{i}^{-1}\right)=l\left(w_{i}\right)+l\left(x_{i}^{-1}\right)$ for all $0 \leq i \leq n-1$.

## Chapter 7 Counting $J$-conjugacy classes in type $A_{n}$

Some work has already been done on counting the number of $W_{J}-W_{K}$ double cosets in a Weyl group $W$ of type $A$, for $J$ and $K$ subsets of $S$. For instance, James and Kerber in [JK] show they correspond to certain $n \times n$ matrices, and Andrew Jones in [Jo] developes a method for counting these. However it seems nothing has been done on counting actual $J$-conjugacy classes.

There are two obvious approaches. One is to take each $W_{J}-W_{J}$ double coset and count the $J$-conjugacy classes contained in it. The other is to take a conjugacy class, and count the $J$-conjugacy classes in it. Since we can count either the number of double cosets, or the number of conjugacy classes, both these would yield a solution. In the following, we take the latter approach, to develop an easy correspondence between $J$-conjugacy classes, and certain placings of numbers in a cycle type. In the second section, we make an attempt to find a formula for this number.

### 7.1. A CORRESPONDENCE

We now turn our attention to $J$-conjugacy classes of $W\left(A_{n}\right)$. We will start with $J=\left\{s_{1}, \ldots, s_{j}\right\}$, although this should be easily generalised to arbitrary $J \subseteq S$.
(7.1.1) Lemma. The $\left\{s_{1}, \ldots, s_{j}\right\}$-conjugacy classes contained in a conjugacy class $C$ of $W\left(A_{n}\right)$ correspond to distinct placings of $\{j+2, \ldots, n+1\}$ in the cycle structure of $C$, up to permutation of cycles, and cyclic permutation within cycles. Proof. The elements of the conjugacy class $C$ correspond to distinct placings of $\{1, \ldots, n+1\}$ in the cycle structure corresponding to $C$. When we refer to distinct placings in a cycle structure we naturally refer to cyclically distinct placings. That is, placement up to cyclical permutation of entries within a cycle, and up to permutation of cycles of same length. So for example, (123) $\equiv(231) \not \equiv(213)$, and $(12)(34) \equiv(34)(12)$.

Those elements of $C$ conjugate under a generator $s_{i}$ are those whose placings of $\{1,2, \ldots, i-1, i+2, \ldots, n+1\}$ are the same. This is since the effect of conjugating
by $s_{i}$ is to exchange the letters $i$ and $i+1$ in the cycle structure, fixing all else.
Conjugating an element by words in $\left\langle s_{1}, \ldots, s_{j}\right\rangle$ has the effect of permuting the letters $\{1, \ldots, j+1\}$ in the cycle structure, and each permutation of $\{1, \ldots, j+1\}$ in the cycle structure of an element corresponds to conjugation by an element of $\left\langle s_{1}, \ldots, s_{j}\right\rangle$.

Thus all elements of $C$ with the same placement of $\{j+2, \ldots, n+1\}$ are conjugate under $\left\langle s_{1}, \ldots, s_{j}\right\rangle$. That is, all distinct placements of $\{1, \ldots, n+1\}$ which have the letters $\{j+2, \ldots, n+1\}$ in the same positions (cyclically the same), are in the same $\left\{s_{1}, \ldots, s_{j}\right\}$-conjugacy class.

This generalizes easily to any $J$-conjugacy class, not just for $J$ principal as above.
(7.1.2) Example. Enumerating the $\left\{s_{1}, s_{2}\right\}$-conjugacy classes in type $W\left(A_{4}\right)$.

Here $j=2, n=4$. We need to count the number of distinct ways of placing $\{j+2, \ldots, n+1\}=\{4,5\}$ in the cycles corresponding to partitions of $n+1=5$. There are seven partitions of $5:(5),(4,1),(3,2),(3,1,1),(2,2,1),(2,1,1,1)$, and $(1,1,1,1,1)$. The following are all the cyclically distinct ways of placing 4 and 5 in the cycle structures corresponding to these seven partitions of 5 (the fullstops correspond to the unfilled places):

$$
\begin{aligned}
& (5) \rightarrow(45 \ldots) \quad(3,2) \rightarrow(45 .)(. .) \\
& \text { (4.5..) } \\
& \text { (4..5.) } \\
& \text { (4...5) } \\
& (4,1) \rightarrow(45 \ldots)(.) \\
& \text { (4.5.)(.) } \\
& \text { (4..5)(.) } \\
& \text { (4...)(5) } \\
& \begin{aligned}
& \\
&\left(3,1^{2}\right)(5 \ldots(45 .)(.)(.)
\end{aligned} \\
& \text { (4.5)(.)(.) } \\
& \text { (4..)(5)(.) } \\
& \text { (5..)(4)(.) } \\
& \text { (4.5)(..) } \\
& \text { (4..)(5.) } \\
& \text { (5..)(4.) } \\
& \left(2,1^{3}\right) \quad \rightarrow \quad(45)(.)(.)(.) \\
& \text { (...)(4)(5) } \\
& \left(1^{5}\right) \rightarrow(4)(5)(.)(.)(.)
\end{aligned}
$$

Thus there are $28\left\{s_{1}, s_{2}\right\}$-conjugacy classes in the Weyl group of type $A_{4}$. One can find a representative of a given $\left\{s_{1}, s_{2}\right\}$-conjugacy class by placing 1,2 , and 3
in the vacant places of the relevant cycle type. Similarly one can list all elements of the $\left\{s_{1}, s_{2}\right\}$-conjugacy class by listing all distinct placings of 1,2 , and 3 in the relevant cycle type.

### 7.2. Some counting

We now turn to the question of counting the number of $J$-conjugacy classes in a given Weyl group of type $A_{n}$, where $J=\left\{s_{1}, \ldots, s_{j}\right\}$.

We start by restricting our attention to an arbitrary conjugacy class $C_{\alpha}$ of $W\left(A_{n}\right)$. As shown in (7.1.1), the number of $J$-conjugacy classes in $C_{\alpha}$, is the same as the number of distinct placings of $n-j$ letters in the cycle structure corresponding to the partition $\alpha$. For convenience and by abuse of notation, we will refer to the cycle structure corresponding to $\alpha$ simply as the cycle structure $\alpha$.

Write $\alpha=\left(\alpha_{1}, \ldots, \alpha_{m}\right)$. Firstly we consider how to distribute the $n-j$ letters in $\alpha$. Any such distribution is in one-to-one correspondence with an $m$-composition of $n-k \lambda \models_{m} n-k$ such that $\lambda \leq \alpha$. By $m$-composition, we mean a composition with $m$ parts, some of which may be zero. A composition $\lambda$ is $\leq \alpha$ if and only if $\lambda_{i} \leq \alpha_{i}$, for all $i$.

Given $\lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right) \models_{m} n-k$, we need to choose exactly which $\lambda_{1}$ of the $n-k$ letters goes into the $\alpha_{1}$-cycle, and which $\lambda_{2}$ of the $n-k-\lambda_{1}$ remaining letters goes into the $\alpha_{2}$-cycle. If we choose in this way, for each $\lambda_{i}$ there are $\left({ }_{i}+\cdots+\lambda_{m}\right)$ ways to pick the $\lambda_{i}$ letters for the $\alpha_{i}$-cycle.

Having chosen the $\lambda_{i}$ letters, we now have to distribute them in the $\alpha_{i}$-cycle. This is the same as the problem of distributing $\lambda_{i}$ letters around a circle of $\alpha_{i}$ places - the answer to which is $\binom{\alpha_{i}-1}{\lambda_{i}-1}$.

So for each $\alpha_{i}$ and $\lambda_{i}$, we have $\binom{\lambda_{i}+\cdots+\lambda_{m}}{\lambda_{i}}\binom{\alpha_{i}-1}{\lambda_{i}-1}$ ways of choosing the $\lambda_{i}$ letters and placing them in the $\alpha_{i}$-cycle.

This gives, for each $\lambda \models_{m} n-k, \lambda \leq \alpha$, we have at most

$$
\prod_{i=1}^{m}\binom{\lambda_{i}+\cdots+\lambda_{m}}{\lambda_{i}}\binom{\alpha_{i}-1}{\lambda_{i}-1}=\frac{(n-k)!}{\lambda_{1}!\ldots \lambda_{m}!} \prod_{i=1}^{m}\binom{\alpha_{i}-1}{\lambda_{i}-1}
$$

ways to place the $n-k$ letters in the cycle structure $\alpha$. However, there is some
repetition of counting here, which we will address soon, but first let us sum over the different $m$-compositions $\lambda$ to give

$$
\sum_{\substack{\lambda \neq m n-k \\ \lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right) \leq \alpha}}\left[\prod_{i=1}^{m}\binom{\lambda_{i}+\cdots+\lambda_{m}}{\lambda_{i}}\binom{\alpha_{i}-1}{\lambda_{i}-1}\right]
$$

$$
=\sum_{\substack{\lambda \mid=m n-k \\ \lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right) \leq \alpha}} \frac{(n-k)!}{\lambda_{1}!\ldots \lambda_{m}!}\left[\prod_{i=1}^{m}\binom{\alpha_{i}-1}{\lambda_{i}-1}\right] .
$$

Now to see where we have duplicated our counting. Consider the case where $\alpha_{i}=\alpha_{i+1}$. Then swapping the entries from $\lambda\left(\lambda_{i}\right.$ and $\left.\lambda_{i+1}\right)$ will give the same cycle, although it will be counted differently in the above sum. Except, that is, when both $\lambda_{i}$ and $\lambda_{i+1}$ are zero - in this case swapping the two is counted as the same in the above product and there is no problem.

In more generality, if the multiplicity of $\alpha_{i}$ in $\alpha$ is $\nu_{i}$, we will need to divide the above product by $\nu_{i}$ ! for each different multiplicity, and then multiply by the factorial of the number of zeros in the set $\left\{\lambda_{i}, \ldots, \lambda_{i+\nu_{i}-1}\right\}$. Let $z\left(\alpha_{i}, \lambda\right)$ be the number of zeros in $\left\{\lambda_{i}, \ldots, \lambda_{i+\nu_{i}-1}\right\}$, and write $\alpha=\left(\alpha_{j}^{\nu_{j}}\right)_{j=1}^{m^{\prime}}$, meaning that there are $m^{\prime}$ distinct entries in $\alpha$, and the multiplicity of each is $\nu_{j}$. Then we have the final result that the number of $J$-conjugacy classes in the conjugacy class $C_{\alpha}$ is given by

$$
\sum_{\substack{\lambda \neq m n-k \\ \lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right) \leq \alpha}} \frac{(n-k)!}{\lambda_{1}!\ldots \lambda_{m}!}\left[\prod_{j=1}^{m^{\prime}} \frac{z\left(\alpha_{j}, \lambda\right)!}{\nu_{j}!}\right]\left[\prod_{i=1}^{m}\binom{\alpha_{i}-1}{\lambda_{i}-1}\right] .
$$

We have thus shown that
(7.2.1) Theorem. Let $J=\left\{s_{1}, \ldots, s_{k}\right\}$. The number of $J$-conjugacy classes in $W\left(A_{n}\right)$ is

$$
\sum_{\alpha \vdash n+1} \sum_{\substack{\begin{subarray}{c}{1=m^{n-k} \\
\lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right) \leq \alpha} }}\end{subarray}} \frac{(n-k)!}{\lambda_{1}!\ldots \lambda_{m}!}\left[\prod_{j=1}^{m^{\prime}} \frac{z\left(\alpha_{j}, \lambda\right)!}{\nu_{j}!}\right]\left[\prod_{i=1}^{m}\binom{\alpha_{i}-1}{\lambda_{i}-1}\right] .
$$

Clearly, if this formula ever is to have a use, it will not be for hand-computation. In small cases, it is easier to make a list using Lemma (7.1.1). It does nevertheless, provide a way to count $J$-conjugacy classes if you can find out all the $m$-compositions $\lambda \leq \alpha$ for each partition $\alpha$.

We can verify the theorem for a case from example (7.1.2):
(7.2.2) Example. Let $n=4$, and $k=2$, as in (7.1.2). Let us consider the case $\alpha=(3,1,1)$. We need to list all the 3 -compositions of $n-k=2$ less than $\alpha$. They are $\lambda_{a}=(2,0,0), \lambda_{b}=(1,1,0), \lambda_{c}=(1,0,1)$, and $\lambda_{d}=(0,1,1)$.

For $\lambda_{a}, \lambda_{a 1}!\lambda_{a 2}!\lambda_{a 3}!=2!0!0!=2$. So the first term of the product for $\lambda_{a}$ is $2!/ 2=1$. We may write $\alpha=\left(3^{1}, 1^{2}\right)$, and $\lambda_{a}=\left(2^{1}, 0^{2}\right)$. So $z\left(\alpha_{1}, \lambda_{a}\right)=0$, and $z\left(\alpha_{2}, \lambda_{a}\right)=2$, and $\nu_{1}=1, \nu_{2}=2$. So the second term of the product for $\lambda_{a}$ is $(0!2!) /(1!2!)=1$. The third term is $\binom{2}{1}\binom{0}{-1}\binom{0}{-1}=2$. This gives for $\lambda_{a}$ a total of 1.1.2 $=2$.

For $\lambda_{b}=(1,1,0)$ we have a first term of $2!/ 1!0!1!=2$. Again $z\left(\alpha_{1}, \lambda_{b}\right)=0$ but $z\left(\alpha_{2}, \lambda_{b}\right)=1$, so the second term is $0!1!/ 1!2!=1 / 2$. The third term is $\binom{2}{0}\binom{0}{0}\binom{0}{-1}=1$, giving a product for $\lambda_{b}$ of 1 . The product for $\lambda_{c}$ is the same essentially, as we have the same $z\left(\alpha_{j}, \lambda\right)$, and the same first and third products.

Finally $\lambda_{d}$ gives us a first term of $2!/ 0!1!1!=2$. Here $z\left(\alpha_{1}, \lambda\right)=1$ and $z\left(\alpha_{2}, \lambda\right)=$ 0 , so the second term is $1!0!/ 1!2!=1 / 2$. Lastly the third term is $\binom{2}{-1}\binom{0}{0}\binom{0}{0}=1$, so our product for $\lambda_{d}$ is 1 .

This gives us the sum for $\alpha=(3,1,1)$ of $2+1+1+1=5$, which is confirmed by our result in (7.1.2).

## Appendix A <br> Alternative proofs of the existence of the minimal basis

In this appendix we present proof of the existence of the minimal basis using the results of Jones [J2] and Geck and Rouquier [GR]. That is, the following methods are an alternative to those in section 3.2, although for the reasons outlined in the introduction, the methods in 3.2 are more general and elementary.

## A. 1 Type $A_{n}$ using norms

Because of the bijection between partitions of $n+1$ and conjugacy classes in type $A_{n}$, we will refer to the conjugacy classes of $W\left(A_{n}\right)$ by $C_{\alpha}$, for $\alpha \vdash n+1$. We denote the length of the shortest element(s) in $C_{\alpha}$ by $l_{\alpha}$.

In [DJ1], the following definition of length is shown to be equivalent to ours:
(A.1.1) Lemma. Let $w \in W$. Then $l(w)=\mid\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i<j$ and $i w>j w\} \mid$.

Let $\pi$ be a $k$-cycle, containing integers $\pi_{1}, \ldots, \pi_{k}$, with $\pi_{1}<\cdots<\pi_{k}$. We define the set

$$
P(\pi)=\left\{(i, j) \in\left\{\pi_{1}, \ldots, \pi_{k}\right\} \times\left\{\pi_{1}, \ldots, \pi_{k}\right\} \mid i<j \text { and } i \pi>j \pi\right\}
$$

and for any $w \in W$ we define,

$$
\mathcal{P}(w)=\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i<j \text { and } i w>j w\} .
$$

Then by Lemma (A.1.1), $l(w)=|\mathcal{P}(w)|$.
(A.1.2) Lemma. Let $\pi$ be a $k$-cycle, containing integers $\pi_{1}, \ldots, \pi_{k}$, with $\pi_{1}<$ $\cdots<\pi_{k}$. Then $|P(\pi)| \geq k-1$.

Proof. The map which identifies $\pi_{i}$ with $i$ takes $\pi$ to a coxeter element of $S_{k}$, and preserves the inequalities between pairs of elements. Any coxeter element of $S_{k}$ has length $\geq k-1$, and so has $\geq k-1$ pairs satisfying the requirements of the
lemma. The inverse image of these pairs provide the required number of pairs for $\pi$. This proves the lemma.

The following lemma is an easy consequence of (A.1.2).
(A.1.3) Lemma. Let $w=\pi_{1} \pi_{2} \ldots \pi_{t}$ be a product of disjoint cycles $\pi_{i}$. Then $l(w) \geq \sum_{i=1, \ldots, t}\left|P\left(\pi_{i}\right)\right|$.

Proof. Recall that $l(w)=\mid\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i<j$ and $i w>j w\} \mid$. Then for each $\pi_{i}$ a disjoint cycle in $w, P\left(\pi_{i}\right)$ will be a subset of this set, and each $P\left(\pi_{i}\right)$ will have pairwise trivial intersection with any other $P\left(\pi_{j}\right)$. So $\mathcal{P}(w)$ will contain the (disjoint) union of all $P\left(\pi_{i}\right)$.
(A.1.4) Lemma. Let $w$ be a shortest element in its conjugacy class $C_{\lambda}$. We can write $w$ as a product of disjoint cycles, $w=\pi_{1} \ldots \pi_{t}$. Then

$$
l(w)=\sum_{i=1, \ldots, t}\left|P\left(\pi_{i}\right)\right|
$$

Proof. Consider the Coxeter element $w_{\lambda}$ in $C_{\lambda}$. Suppose $\lambda=\left(\lambda_{1}, \ldots, \lambda_{t}\right)$. Then $w_{\lambda}=\left(12 \ldots \lambda_{1}\right)\left(\lambda_{1}+1 \ldots \lambda_{1}+\lambda_{2}\right)(\ldots) \ldots$ is a shortest element of $C$. Clearly there are no pairs $(i, j) \in \mathbb{N} \times \mathbb{N}$ such that $i<j$ and $i w_{\lambda}>j w_{\lambda}$ when $i$ and $j$ are from different cycles of $w_{\lambda}$.

Thus $\left\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i<j\right.$ and $\left.i w_{\lambda}>j w_{\lambda}\right\}$ is the disjoint union of the sets of pairs of the disjoint cycles in $w_{\lambda}$, and so the lemma is proved for $w_{\lambda}$.

For any other shortest element $w=\pi_{1} \ldots \pi_{t}\left(\pi_{i}\right.$ disjoint cycles of length $\left.\lambda_{i}\right)$,

$$
\begin{aligned}
l\left(w_{\lambda}\right) & =\sum_{i=1, \ldots, t}\left(\lambda_{i}-1\right) \\
& \leq \sum_{i=1, \ldots, t}\left|P\left(\pi_{i}\right)\right| \\
& \leq l(w) \\
& =l\left(w_{\lambda}\right)
\end{aligned}
$$

and the lemma is proved.
(A.1.5) Lemma. Let $w$ be a shortest element in $C_{\lambda}$. Then each disjoint cycle of $w$ contains only consecutive integers.

Proof. Suppose $w$ had a cycle whose integer entries were not consecutive. Denote the entries of this cycle $a_{1}, a_{2}, \ldots, a_{p}$, with $a_{i}<a_{i+1}$ for all $1 \leq i \leq p-1$, and suppose there exists integers $k$ and $j$ such that $a_{j}<k<a_{j+1}$.

In a cycle such as this, there must exist an element of $\left\{a_{1}, \ldots, a_{j}\right\}$ which is sent to $\left\{a_{j+1}, \ldots, a_{p}\right\}$, and an element of $\left\{a_{j+1}, \ldots, a_{p}\right\}$ which is sent back - otherwise the cycle would split into two separate cycles.

The integer $k$ is in a cycle sending it either to an integer $\geq k$ or $\leq k$ (with equality if it is in a 1-cycle).

Suppose that it is sent to a number less than $k$. Then as there is a number $x$ in $\left\{a_{1}, \ldots, a_{j}\right\}$ which is sent to greater than $k$, the pair $(x, k) \in \mathcal{P}(w)$. Then $l(w)$ is at least one longer than minimal length for $C_{\lambda}$, and $w$ cannot be minimal. Alternatively, if $k$ were sent to greater than $k$, then if $y$ is an integer in $\left\{a_{j+1}, \ldots, a_{p}\right\}$ which is sent to less than $k$, then the pair $(k, y)$ would satisfy our requirements, and the same conclusion is reached.

We say that $w$ corresponds to a composition $\mu=\left(\mu_{1}, \mu_{2}, \ldots\right)$ if in cycle notation the integers $1, \ldots, \mu_{1}$ appear in a cycle of their own, the integers $\mu_{1}+1, \ldots, \mu_{1}+\mu_{2}$ appear in a cycle of their own, and so on.

So we have shown that if $w$ is shortest in its conjugacy class, then all its cycles contain consecutive integers. Thus we have that the set of shortest elements in $C$ can be partitioned into subsets, each corresponding to a different composition of $n+1$. We now claim that for each composition $\mu$ of $n+1$, the shortest elements corresponding to $\mu$ are in the same equivalence class under $\sim_{s}$.
(A.1.6) Lemma. The shortest elements corresponding to the composition $\mu$ are in the same equivalence class under $\sim_{s}$.

Proof. We can reach all Weyl group elements corresponding to the composition $\mu=\left(\mu_{1}, \ldots, \mu_{t}\right)$ by conjugating by elements of the parabolic subgroup $S_{\mu_{1}} \times$ $\cdots \times S_{\mu_{t}}$. Since these summands commute, it suffices to consider the shortest elements obtained by conjugating a single $\mu_{i}$-cycle by simple reflections from the
corresponding summand. But these summands correspond to Coxeter elements of Young subgroups, which can easily be shown to be equivalent under $\sim_{s}$, using the above lemma.

Dipper and James proved the following result in [DJ2; 2.11].
(A.1.7) Lemma. Let $w$ and $u$ be shortest elements of $C$ corresponding to different compositions of $n+1$, and let $h=\sum_{w \in W} f_{w}(\xi) \tilde{T}_{w} \in Z(\mathcal{H})$. Then $f_{w}(\xi)=f_{u}(\xi)$.

We then have the following:
(A.1.8) Proposition. Shortest elements from the same conjugacy class have the same coefficient in a central element of $\mathcal{H}$.

Proof. We have shown that all shortest elements correspond to a composition of $n+1$. For each composition, we have shown (Lemma (A.1.6)) that the shortest elements are equivalent under $\sim_{s}$, and so have the same coefficient. Dipper and James' result (Lemma (A.1.7)) tells us that shortest elements from different compositions also have the same coefficient in a central element. The theorem follows.
(A.1.9) Lemma. Suppose there exists a $\Gamma_{C}$ for $C \in \operatorname{ccl}(W)$. If $r \tilde{T}_{w} \leq h \in Z(\mathcal{H})$ for $w \in C_{\min }$, then $r \Gamma_{C} \leq h$ also.

Proof. By the above lemma, $r \tilde{T}_{C_{\text {min }}} \leq h$. Using the same methods as (3.3.3), we have that the algorithm $\mathfrak{A}$ is well-defined when begun on $r \tilde{T}_{C_{\text {min }}}$ (we can use $r \Gamma_{C}$ as an upper bound). Then, by (3.1.2), all the additions under $\mathfrak{A}$ are in fact implications for $r \tilde{T}_{C_{\text {min }}} \leq h$.
(A.1.10) Lemma. If there exists an element $L_{C}$ satisfying the following properties:
(1) $\left.L_{C}\right|_{\xi=0}=a \tilde{T}_{C}$, and
(2) $L_{C}-a \tilde{T}_{C}$ contains no shortest elements of any conjugacy class, then $\mathfrak{A}$ is well-defined on $\tilde{T}_{C}$ or $\tilde{T}_{C_{\text {min }}}$.

Proof. Using the same methods as (3.3.3), we see that $\mathfrak{A}$ is well defined when started on $a \tilde{T}_{C}$, and ends with $L_{C}$. Consequently, every term of $L_{C}$ has a factor
of $a$. Thus we may divide $L_{C}$ by $a$, obtaining a central element satisfying (3.2.1.1) and (3.2.1.2). Then (3.3.5) gives us the result.

We now show the existence in type $A_{n}$ of the class elements, using results of Lenny Jones from [J2]. Despite the new work of Geck and Rouquier ([GR]) described in the next section, this proof in type $A_{n}$ remains interesting as it is entirely elementary and does not need character theory.

The following definition is attributed in [J2] to Hoefsmit and Scott:
(A.1.11) Definition. Let $W^{\prime}$ be a parabolic subgroup of $W$, and $\mathfrak{D}$ be the set of distinguished right coset representatives of $W^{\prime}$ in $W$. For $h \in \mathcal{H}$, we define the relative norm of $h$ to be

$$
N_{W, W^{\prime}}(h)=\sum_{d \in \mathfrak{D}} \tilde{T}_{d^{-1}} h \tilde{T}_{d} .
$$

The following lemma is vital for the Jones results (see [J2;(2.13)]):
(A.1.12) Lemma. If $h \in Z_{\mathcal{H}}\left(\mathcal{H}\left(W^{\prime}\right)\right)$, then $N_{W, W^{\prime}}(h) \in Z(\mathcal{H})$.

A parabolic subgroup $W_{\lambda}$ of $W$ corresponds to a partition $\lambda$ of $n+1$ in the following way. If $\lambda=\left(\lambda_{1}, \ldots, \lambda_{r}\right) \vdash n+1$, then we define

$$
W_{\lambda}=W_{\lambda_{1}} \times \cdots \times W_{\lambda_{r}}
$$

where

$$
W_{\lambda_{i}}=\left\langle s_{\lambda_{1}+\cdots+\lambda_{i-1}+1}, \ldots, s_{\lambda_{1}+\cdots+\lambda_{i}-1}\right\rangle .
$$

[Note that there are many compositions of $n+1$ corresponding to the same partition, each reflecting a permutation of the $\lambda_{i}$ 's. These different compositions give different (conjugate) parabolic subgroups, but we will focus on the "standard" representative corresponding to the ordering of the $\lambda_{i} \mathrm{~s}$ which is a partition.]

We denote the Iwahori-Hecke algebra corresponding to the parabolic subgroup $W_{\lambda}$ by $\mathcal{H}_{\lambda}:=\mathcal{H}\left(W_{\lambda}\right)$.

For such a component subgroup $W_{\lambda_{i}}$ we define $w_{\lambda_{i}}$ to be its Coxeter element $w_{\lambda_{i}}=s_{\lambda_{1}+\cdots+\lambda_{i-1}+1} \ldots s_{\lambda_{1}+\cdots+\lambda_{i}-1}$, and write $l_{\lambda}=\sum_{i=1}^{r}\left(\lambda_{i}-1\right)$ for the length
of the Coxeter element of $W_{\lambda}$. There is also a corresponding conjugacy class $C_{\lambda}$ of $W$ for each partition $\lambda$ of $n+1$, for which the Coxeter element $w_{\lambda_{1}} \ldots w_{\lambda_{r}}$ of $W_{\lambda}$ is a shortest representative.

Then for any partition $\lambda \vdash n+1$, let

$$
\eta_{\lambda}=\prod_{i=1}^{r} N_{W_{\lambda_{i}-1}, 1}\left(\tilde{T}_{w_{\lambda_{i}}}\right)
$$

and

$$
b_{\lambda}=N_{W, W_{\lambda}}\left(\eta_{\lambda}\right)
$$

The main result of [J2] is that the set of $b_{\lambda}$ for $\lambda \vdash n+1$ is a $\mathbb{Q}[\xi]$-basis for $Z(\mathcal{H})$. Our main interest in these elements lies in some of the properties they have in addition to being a basis. The following properties are largely proven in [J2] at various points in the paper:
(A.1.13) Proposition. Let $w \in W$ and $0 \neq r \in \mathbb{N}[\xi]$. Then
(i) $\eta_{\lambda} \in Z\left(\mathcal{H}_{\lambda}\right)$,
(ii) $b_{\lambda} \in Z(\mathcal{H})^{+}$,
(iii) If $r \tilde{T}_{w} \leq b_{\lambda}$, then $l(w) \geq l_{\lambda}$,
(iv) If $r \tilde{T}_{w} \leq b_{\lambda}$ and $l(w)=l_{\lambda}$, then $w \in C_{\lambda, \min }$ and $r \in \mathbb{N}$,
(v) $\left.b_{\lambda}\right|_{\xi=0}=a \tilde{T}_{C_{\lambda}}$ for some $a \in \mathbb{N}$.

Proof. (i) [J2;(3.23)].
(ii) The centrality of $b_{\lambda}$ follows from (A.1.12) and (i), and the positivity follows since $b_{\lambda}$ is a sum of products of sums of products of positive elements, and so is also positive, by (2.1.1).
(iii) [J2;(3.25)].
(iv) [J2;(3.27),(3.28)].
(v) $[J 2 ;(3.29)]$.
(A.1.14) Theorem. For each conjugacy class $C$ of $W=W\left(A_{n}\right)$ there exist elements $L_{C} \in Z(\mathcal{H})^{+}$satisfying (A.1.10)(1) and (2).

Proof. We proceed by reverse induction on the length $l_{\lambda}$ of the shortest element of the conjugacy class $C_{\lambda}$.

The longest shortest element of any conjugacy class is that of the Coxeter class, which corresponds to the partition $\lambda=(n+1)$. In this case we have that $b_{(n+1)}$ satisfies (1) directly from (A.1.13)(v). By (A.1.13)(iii) and (iv), the only elements of length $n$ are those from $C_{(n+1), \min }$, and these appear with integer coefficient in $b_{(n+1)}$. Then by (A.1.13)(v), they in fact all have the same coefficient. All other terms in $b_{(n+1)}$ are strictly longer than $n$, and not shortest in any conjugacy class, and so $b_{(n+1)}$ also satisfies (2), and we have the existence of the required $L_{C}$.

Suppose inductively that for $l_{\lambda}>k$ we have the existence of $L_{C_{\lambda}}$ with the required properties. Then we also have that $\Gamma_{C_{\lambda}}$ exists for $l_{\lambda}>k$, and thus we have the results of section 3.3 - the well-definedness of the algroithm $\mathfrak{A}$ - for those conjugacy classes.

Take a conjugacy class $C_{\lambda}$ with $l_{\lambda}=k$. The only terms of length $k$ in the Jones element $b_{\lambda}$ are elements of $C_{\lambda, \min }$. If there are shortest elements of other conjugacy classes in $b_{\lambda}$ they must have length strictly greater than $k$. By (A.1.9), if $r \tilde{T}_{w} \leq b_{\lambda}$ for $w \in C_{\min }$ for some $C$, then $r \Gamma_{C} \leq b_{\lambda}$, and so we may subtract $r \Gamma_{C}$ from $b_{\lambda}$ while remaining in $Z(\mathcal{H})^{+}$. In this way we may remove all shortest elements other than those of length $k$ from $b_{\lambda}$, giving us a positive central element which specializes to an $\mathbb{N}$-multiple of $\tilde{T}_{C_{\lambda}}$ and which contains no other shortest elements with non-zero coefiicient. In other words, we have an element satisfying (1) and (2). This proves the theorem.
(A.1.15) Corollary. In type $A_{n}$ we have the following:
(i) For any $i \in \mathbb{N}$, $\mathfrak{A}^{i}\left(\tilde{T}_{C}\right)$ and $\mathfrak{A}^{i}\left(\tilde{T}_{C_{\text {min }}}\right)$ are well-defined, and there exists finite $n, n^{\prime} \in \mathbb{N}$ such that $\mathfrak{A}^{n}\left(\tilde{T}_{C}\right)=\mathfrak{A}^{n^{\prime}}\left(\tilde{T}_{C_{\text {min }}}\right)=\Gamma_{C}$ satisfies (3.2.1.1) and (3.2.1.2).
(ii) $Z(\mathcal{H})_{\min }^{+}=\left\{\Gamma_{C} \mid C \in \operatorname{ccl}(W)\right\}$.
(iii) $Z(\mathcal{H})_{\min }^{+}$is an $R$-basis for $Z(\mathcal{H})$.

Proof. (i) follows from (A.1.10) and (A.1.12).
(ii) follows from (i) and (3.4.2).
(iii) follows from (ii) and (3.4.3).

## A. 2 Other crystallographic types using characters

Recent results of Geck and Rouquier ([GR]) provide the type of element required
for the results in section 3.3 in order to prove the algorithm is well-defined for all types of Weyl group, and that the primitive minimal positive central elements are a basis for $Z(\mathcal{H})$.

Define the irreducible characters $\phi_{i}: \mathcal{H}_{\mathbb{Q}(\xi)} \rightarrow \mathbb{Q}(\xi)$. These are $\mathbb{Q}(\xi)$-linear maps with the property that for any $a, b \in \mathcal{H}_{\mathbb{Q}(\xi)}, \phi_{i}(a b)=\phi_{i}(b a)$. Starkey and Ram (independently in [C2] and [R] respectively) have shown in type $A_{n}$ that these characters are constant on shortest elements of conjugacy classes. Geck and Pfeiffer extended this to all types of finite Weyl group in [GP] using (1.1.2).

Since $\mathcal{H}_{\mathbb{Q}(\xi)}$ is isomorphic to $\mathbb{Q}(\xi) W$, we have that the number of irreducible characters is the same as the number of conjugacy classes. Theorem (1.1.2) shows that we may write the image of any generator $\tilde{T}_{w}$ under any central function $\phi$ as an $\mathbb{N}[\xi]$-linear combination of images of shortest elements from conjugacy classes, using the relations $\phi\left(\tilde{T}_{s d s}\right)=\phi\left(\tilde{T}_{s d} \tilde{T}_{s}\right)=\phi\left(\tilde{T}_{s} \tilde{T}_{s d}\right)=\phi\left(\tilde{T}_{d}+\xi \tilde{T}_{s d}\right)=$ $\phi\left(\tilde{T}_{d}\right)+\xi \phi\left(\tilde{T}_{s d}\right)$ and $\phi\left(\tilde{T}_{d s}\right)=\phi\left(\tilde{T}_{s d}\right)$ for $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}$. Thus we may write

$$
\phi_{i}\left(\tilde{T}_{y}\right)=\sum_{C} f_{y, C} \phi_{i}\left(\tilde{T}_{w_{C}}\right)
$$

for any $y \in W$, all irreducible characters $\phi_{i}$, and for some $w_{C} \in C_{\text {min }}$. Further, the $f_{y, C}$ are unique since the irreducible characters are linearly independent (which means the matrix $\left(\phi_{i}\left(\tilde{T}_{w_{C_{j}}}\right)\right)$ is invertible $)$.

Geck and Rouquier ([GR;Sect. 5]) then point out that the functions $f_{C}: \mathcal{H} \rightarrow$ $\mathbb{Z}[\xi]$ defined on the generators by sending $\tilde{T}_{w}$ to $f_{w, C}$ are central, and that for any central function $\varphi \in C F(\mathcal{H})$ we may write $\varphi=\sum_{C} \varphi\left(\tilde{T}_{w_{C}}\right) f_{C}$, which means the set $\left\{f_{C}: C \in \operatorname{ccl}(W)\right\}$ is a $\mathbb{Z}[\xi]$-basis for $C F(\mathcal{H})$. They then call upon the correspondence between central functions and central elements to get elements $z_{C}$ which form a $\mathbb{Z}[\xi]$-basis for the centre of $\mathcal{H}, z_{C}=\sum_{w \in W} f_{C}\left(\tilde{T}_{w}\right) \tilde{T}_{w^{-1}}$. [Note that since Geck and Rouquier work over $\mathbb{Z}\left[q, q^{-1}\right]$, they need an additional weighting factor $q^{-l(w)}$ which is not necessary over $\mathbb{Z}[\xi]$.] A key part of their proof is the recognition that for $w_{C^{\prime}} \in C_{\min }^{\prime}, f_{w_{C^{\prime}}, C}=\delta_{C, C^{\prime}}$.

Our methods then provide an alternative proof that this set of elements is a basis for the centre:
(A.2.1) Lemma. The set $\left\{\sum_{w \in W} f_{C}\left(\tilde{T}_{w}\right) \tilde{T}_{w} \mid C \in \operatorname{ccl}(W)\right\}$ satisfies properties (3.2.1.1) and (3.2.1.2), and so $z_{C}=\Gamma_{C}$ for all $C$.

Proof. The centrality follows directly from (3.1.1) and the relations $\phi\left(\tilde{T}_{s d s}\right)=$ $\phi\left(\tilde{T}_{d}\right)+\xi \phi\left(\tilde{T}_{d s}\right)$ and $\phi\left(\tilde{T}_{d s}\right)=\phi\left(\tilde{T}_{s d}\right)$ for any $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}$ and any central function $\phi$.

The image of a shortest element of a conjugacy class under $f_{C}$ is either one or zero, so we have that the coefficient of a shortest element $\tilde{T}_{w_{C^{\prime}}}$ in $\sum_{w \in W} f_{C}\left(\tilde{T}_{w}\right) \tilde{T}_{w}$ is one if $C^{\prime}=C$ and zero otherwise. Thus (3.2.1.2) is satisfied.

If $w \in C^{\prime}$ for any conjugacy class $C^{\prime}$ of $W$, then $f_{C}\left(\tilde{T}_{w}\right)=f_{C}\left(\tilde{T}_{w_{C^{\prime}}}\right)+\xi X$ where $X$ is an $\mathbb{N}[\xi]$-linear combination of images $f_{C}\left(\tilde{T}_{u}\right)$. So since $f_{C}\left(\tilde{T}_{w_{C^{\prime}}}\right)=0$ unless $C=C^{\prime}$, and one if it does, we have $\left.\sum_{w \in W} f_{C}\left(\tilde{T}_{w}\right) \tilde{T}_{w}\right|_{\xi=0}=\tilde{T}_{C}$, so (3.2.1.1) is satisfied.

Finally the equality $\sum_{w \in W} f_{C}\left(\tilde{T}_{w}\right) \tilde{T}_{w}=\sum_{w \in W} f_{C}\left(\tilde{T}_{w}\right) \tilde{T}_{w^{-1}}$ follows since the latter also satisfies both (3.2.1.1) and (3.2.1.2), and so by (3.2.6) they are the same, and are the element $\Gamma_{C}$.

The existence of these elements then provides us with the means to draw our general conclusion a la (3.4.3):
(A.2.2) Theorem. $Z(\mathcal{H})_{\min }^{+}=\left\{\sum_{w \in W} f_{C}\left(\tilde{T}_{w}\right) \tilde{T}_{w} \mid C \in \operatorname{ccl}(W)\right\}$, and is a $\mathbb{Z}[\xi]$ basis for $Z(\mathcal{H})$.

## Appendix B

Conjugacy classes in type $H_{3}$

There are two conjugacy classes which are just singleton sets:

$$
\{1\}, \text { and }\left\{s_{1} t s_{1} t s_{1} s_{2} s_{1} t s_{1} s_{2} t s_{1} t s_{1} s_{2}\right\}
$$

(the shortest and longest words in $W$ ). There are eight more conjugacy classes, which we display below. We will abbreviate $s_{1}$ to simply 1 , and $s_{2}$ to 2 .

It is interesting to observe the pairing of conjugacy classes in $H_{3}$ - each one has an "inverse", graphically upside down.
$C_{t 12}$ :

$C_{t 1 t 1}$ :

$C_{t}$ :

$C_{t 2}$ :

$C_{t 1}$ :


## $C_{1 t 121 t 1+2}$ :


$C_{12}$ :

$C_{1 t 1 t 2}$ :


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