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The Minimal Basis for the Centre of an Iwahori–Hecke Algebra

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This paper arose out of an attempt to generalize the $\mathbb{Q}[q, q^{-1}]$-basis for the centre of an Iwahori–Hecke algebra $\mathscr{H}_q$ found by Jones to $\mathbb{Z}[q, q^{-1}]$ and to other types. Considering the Iwahori–Hecke algebra $\mathscr{H}$ over a subring $\mathbb{Z}[\xi]$ of $\mathbb{Z}[q^{1/2}, q^{-1/2}]$, where $\xi = q^{1/2} - q^{-1/2}$, we use a new and natural definition of positivity on $\mathscr{H}$ to describe the "minimal" $\mathbb{Z}[\xi]$-basis for $Z(\mathscr{H})$ in terms of a partial order on the positive part of $\mathscr{H}$. The main result is to show that this minimal basis is the set of "primitive" minimal elements of the positive part of the centre, for any Weyl group. In addition, the primitive minimal positive central elements can be characterized as exactly those positive central elements which specialize (on setting $\xi = 0$, the equivalent of setting $q = 1$ in $\mathbb{Z}[q^{1/2}, q^{-1/2}]$) to the sum of elements in a conjugacy class, and which apart from the shortest elements from that conjugacy class sum contain no other terms corresponding to shortest elements of any conjugacy class. A constructive algorithm is provided for obtaining the minimal basis. We use the results of Jones to achieve the result in type A_n without the need for character theory, and give the result for all Weyl groups using the character theoretical results of Geck and Rouquier. Finally we discuss the non-crystallographic cases and give some explicit examples. © 1999 Academic Press

INTRODUCTION

This paper falls into the gènre of work attempting to understand the relationship between the Weyl group algebras and their Iwahori–Hecke algebra counterparts. Key in this area are the results of Tits, Benson–Curtis, and

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Lusztig, proving that the Iwahori–Hecke algebra over a field K is isomorphic to the group algebra of W over K if it is semisimple (see [CR, Sect. 68] or [C1, Sect. 10.11]), and the cell structure of $\mathcal{H}$ defined by Kazhdan and Lusztig (see [KL]) which provided the tools to make the isomorphism explicit (as in [L]).

Specifically here we look at analogies between the structures of their centres. A basis for the centre of $\mathcal{H}$ over K can be easily obtained using the norm structure of Hoefsmit and Scott described by Jones in [J]. This paper of Jones in 1990 also extended this approach to a "relative norm," with which he was able to explicitly describe a basis for the centre of type A_n over $\mathbb{Q}[q, q^{-1}]$.

This paper was something of a breakthrough as little had been known until then about what a basis for the centre might actually look like. However, the real key has turned out to be the Weyl group result of Geck and Pfeiffer [GP], showing one can find a sequence of conjugations by simple reflections (without increasing length) linking any element with a shortest element of its conjugacy class (see Theorem 1.1 below). This enabled them to extend the independent results of Starkey [C2] and Ram [R] from type A_n to all types—they showed that the irreducible characters of $\mathcal{H}$ are constant on shortest elements of a conjugacy class.

Most recently Geck and Rouquier [GR] used this to obtain a $\mathbb{Z}[q, q^{-1}]$-basis for the centre of $\mathcal{H}$ for all Weyl groups, in terms of the irreducible characters of $\mathcal{H}$. The basis we obtain is identical to theirs.

The techniques in this paper are twofold: use of partial order on the positive part of the centre; and use of an elementary understanding of the centralizer of the element corresponding to a simple reflection in $\mathcal{H}$, to define a constructive algorithm to build elements of the centre. The main results are then obtained: that the primitive minimal positive central elements are a $\mathbb{Z}[\xi]$-basis for the centre, and have a very simple characterization; and that the algorithm constructing such elements is well defined. The major part is more-or-less combinatorial—showing that these two results depend only on the existence of certain types of elements. Then we prove the existence of the required elements, adapting in type A_n the basis described by Jones, and in general using the basis found by Geck and Rouquier. Thus the result for type A_n stands independent of any character theory results.

For simplicity we present the material here in the one-parameter case (in general, there will be a different parameter q_s for each generator $s \in S$, with $q_s = q_t$ when s and t are conjugate). The results also hold for the multi-parameter case, and are presented in that generality in [Fr]. The approaches in this paper are also being applied to centralizers of parabolic subalgebras, and to the centres of affine Hecke algebras.

1. PRELIMINARIES

Let W be a Weyl group with generating set S, and length function $l: W \to \mathbb{N}$. Then for $s, s' \in S$, W has relations

$$s^2 = 1$$

$$(ss')^{m_{ss'}} = 1$$

for some $m_{ss'} \in \mathbb{N}$. Each Weyl group is partitioned into conjugacy classes C. Let l_C be the length of the shortest elements in C and let $C_{\min}$ be the set of shortest elements in C. Let $\mathrm{ccl}(W)$ be the set of conjugacy classes in W.

The main Weyl group result which we will need is the following conjugacy theorem from [GP]. First we need to introduce some notation.

For $w, w' \in W$ and $s \in S$, we say $w \longrightarrow_s w'$ if $sws = w'$ and $l(w) \geq l(w')$.

For example, later we will be working with double cosets $\langle s \rangle d \langle s \rangle$ for d distinguished, which consist of elements $\{d, ds, sd, sds\}$ if $ds \neq sd$ or $\{d, ds\}$ if $ds = sd$. In the former case, the only relations within the double coset are $sds \longrightarrow_s d$, $sd \longrightarrow_s ds$, and $ds \longrightarrow_s sd$. There are no non-trivial relations in the case that $ds = sd$, just $d \longrightarrow_s d$ and $ds \longrightarrow_s ds$.

If there is a sequence of elements $w_0, w_1, \ldots, w_n \in W$ such that for each i, $w_i \longrightarrow_{s_i} w_{i+1}$ for some $s_i \in S$, then we simply write $w_0 \to w_n$.

(1.1) THEOREM (Geck–Pfeiffer). *Let $C \in \mathrm{ccl}(W)$. Then for each $w \in C$, there exists $w' \in C_{\min}$ such that $w \to w'$.*

The theorem shows that for any element of a conjugacy class C we may find a sequence of conjugations by simple reflections which never increases in length, and ends with a shortest element of C. Another way to think of the theorem is found in the following corollary. The following equivalence classes are also defined in [GP, Sect. 3] and used to prove (1.1) for the exceptional types.

For any conjugacy class C and $s \in S$ we can define an equivalence relation $\sim_s$ on C by writing $w \sim_s u$ if $sws = u$ and $l(w) = l(u)$. We then define a larger equivalence class $\sim_S$ to be generated by the relations $\sim_s$ for $s \in S$. The $\sim_S$-equivalence classes consist of elements of the same length which can be reached from each other by a finite sequence of conjugations by simple reflections, where each step in the sequence gives an element of the same length.

Each conjugacy class C is the disjoint union of such $\sim_S$-equivalence classes, so we can specify uniquely the class by choosing a representative from it. Thus we denote the $\sim_S$-equivalence class containing w by C^w.

(1.2) COROLLARY. *Let $w \in C \setminus C_{\min}$. Then there exists $u \in C^w$ and $t \in S$ such that $l(tut) = l(u) - 2$.*

This means that in every equivalence class C^w not containing shortest elements, there is at least one element which shortens on conjugation by a simple reflection.

We define the Iwahori–Hecke algebra $\mathcal{H}_q$ corresponding to W to be the associative $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-algebra generated by the set $\{T_s\}_{s\in S}$, with relations

$$T_s^2 = qT_1 + (q-1)T_s \qquad (*)$$

and, if $w = s_1 \cdots s_i$ is a reduced expression for w,

$$T_w = T_{s_1} \cdots T_{s_i}.$$

We will find it useful to change the base ring of $\mathcal{H}_q$ from $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ to $\mathbb{Z}[\xi]$ by setting $\tilde{T}_s = q^{-1/2}T_s$ and $\xi = q^{1/2} - q^{-1/2}$ to give us a $\mathbb{Z}[\xi]$-subalgebra denoted $\mathcal{H}$. The relation $(*)$ in $\mathcal{H}$ then becomes

$$\tilde{T}_s^2 = \tilde{T}_1 + \xi\tilde{T}_s,$$

which has the obvious benefit of being simpler. It also has valuable positivity properties, since for any two basis elements $\tilde{T}_x$ and $\tilde{T}_y \in \mathcal{H}$, their product $\tilde{T}_x\tilde{T}_y = \sum_{w\in W} f_{x,y,w}\tilde{T}_w$ has all coefficients $f_{x,y,w}$ in $\mathbb{N}[\xi]$. Thus the product of any two elements of $\mathcal{H}$ whose coefficients are from $\mathbb{N}[\xi]$ (that is, they are linear combinations of the $\tilde{T}_w$ over $\mathbb{N}[\xi]$) also has coefficients in $\mathbb{N}[\xi]$. These observations motivate the definition of $\mathcal{H}^+$ in the next section.

If X is a subset of W (for example, a conjugacy class), we denote by $\tilde{T}_X$ the sum of generators $\tilde{T}_x$ for $x \in X$. That is,

$$\tilde{T}_X := \sum_{x\in X} \tilde{T}_x.$$

$\mathcal{H}_q$ can be obtained from $\mathcal{H}$ by the following change of coefficient ring:

$$\mathcal{H}_q \cong \mathbb{Z}[q^{1/2}, q^{-1/2}] \otimes_{\mathbb{Z}[\xi]} \mathcal{H}.$$

Consequently the centre of $\mathcal{H}$ embeds in the centre of $\mathcal{H}_q$, and a $\mathbb{Z}[\xi]$-basis of $Z(\mathcal{H})$ will become a $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-basis for $Z(\mathcal{H}_q)$ after changing ξ back to $q^{1/2} - q^{-1/2}$ and $\tilde{T}_w$ back to $q^{-(l(w))/2}T_w$. We deal with this, and with obtaining a $\mathbb{Z}[q, q^{-1}]$-basis from the minimal $\mathbb{Z}[\xi]$-basis, at the end of Section 4.

Frequent use will be made of specializing the parameter ξ to zero (equivalent to setting $q = 1$ in $\mathcal{H}_q$), so for any $h \in \mathcal{H}$, we write $h_0 = h|_{\xi=0}$ for this specialization.

2. THE MAIN THEOREM

We now make a brief excursion into higher generality to define positivity and obtain some basic consequences of the definition.

Let A be a free $\mathbb{Z}[\xi]$-module with basis $X = \{x_1, \dots, x_n\}$ and let A_0 be the free $\mathbb{Z}$-module with the same basis. Then $A = \mathbb{Z}[\xi]X$ and $A_0 = \mathbb{Z}X$.

We may also consider A as a $\mathbb{Z}$-module with basis $\{\xi^i x_j \mid \text{for } i \geq 0 \text{ and } 1 \leq j \leq n\}$. Then we have

$$A = \sum_{i \geq 0} A_0 \xi^i.$$

It would be natural to consider $\mathbb{N}[\xi]$ to be the positive part of the ring $\mathbb{Z}[\xi]$, and there is a similarly natural partial order on the elements of $\mathbb{N}[\xi]$: if $f = \sum_{i \geq 0} f_i \xi^i$ and $g = \sum_{i \geq 0} g_i \xi^i$ for $f_i, g_i \in \mathbb{N}$, then $f \leq g$ if and only if $f_i \leq g_i$ for all i. An equivalent expression of this is to say $f \leq g$ if and only if $g - f \in \mathbb{N}[\xi]$.

This partial order on the positive part of $\mathbb{Z}[\xi]$ induces a partial order on the positive part of any $\mathbb{Z}[\xi]$-module. Since $A = \mathbb{Z}[\xi]X$, define the *positive part of* A to be $A^+ = \mathbb{N}[\xi]X$. Define a partial order on A^+ by saying that if $a = \sum_{x \in X} a_x x$ and $b = \sum_{x \in X} b_x x$, where $a_x, b_x \in \mathbb{Z}[\xi]$, then $a \leq b$ in A^+ if and only if $a_x \leq b_x$ in $\mathbb{N}[\xi]$ for all $x \in X$. This is equivalent to saying $a \leq b$ in A^+ if and only if $b - a \in A^+$.

There is an equally obvious partial order on $A_0^+ = \mathbb{N}X$. If $a_0 = \sum_{x \in X} c_x x$ and $b_0 = \sum_{x \in X} d_x x$ for $a_x, b_x \in \mathbb{N}$ then $a_0 \leq b_0$ in A_0 if and only if $c_x \leq d_x$ in $\mathbb{N}$, which is equivalent to having $b_0 - a_0 \in A_0^+$.

If we were to turn A into a $\mathbb{Z}[\xi]$-algebra by defining a multiplication between its basis elements x_i such that $x_i x_j \in \sum_{i=1}^n \mathbb{N}[\xi] x_i$, then we have the following self-evident lemma:

(2.1) LEMMA. *If $x_i x_j \in \sum_{k=1}^n \mathbb{N}[\xi] x_k$ for all $1 \leq i, j \leq n$, then sums and products of elements of A^+ are also in A^+.*

For any $\mathbb{Z}[\xi]$-submodule B of A, let $\min(B^+)$ be the set of non-zero minimal elements of the partial ordering $(B^+, \leq)$, and similarly let $\min(B_0^+)$ be the set of non-zero minimal elements of the poset $(B_0^+, \leq_0)$.

The elements of $\min(A_0^+)$ are simply the elements of X (which are a $\mathbb{Z}$-basis for A_0), and the elements of $\min(A^+)$ are ξ^i-multiples of elements of X (which are a $\mathbb{Z}$-basis for A). The sets $\min(A^+)$ and $\min(B^+)$ are not finite—for example, if $a \neq 0$ is minimal in B^+, then so is ξa, and so is $\xi^2 a$, and so on. We restrict attention to a set of representatives of $\min(B^+)$ (so as to exclude ξ-multiples) by saying a is *primitive* if

$$a_0 \in \sum_{x_i \in X} \mathbb{N} x_i \qquad \text{and} \qquad a_0 \neq 0.$$

Let $B^+_{\min}$ be the set of primitive minimal elements of the poset $(B^+, \leq)$. In other words, $B^+_{\min}$ is the set of minimal elements of B^+ which do not specialize to zero. We then have

$$\min(B^+) = \bigcup_{i\geq 0} \xi^i B^+_{\min}.$$

In this paper we look at $A = \mathcal{H}$ with $\mathbb{Z}[\xi]$-basis $\{\tilde{T}_w \mid w \in W\}$, and $B = Z(\mathcal{H})$. The multiplication between the elements $\tilde{T}_w \in \mathcal{H}^+$ has the positivity property required for Lemma 2.1, so the conclusion holds: that the sums and products of elements of $\mathcal{H}^+$ are also in $\mathcal{H}^+$. This is a simple yet significant benefit of moving to the ring $\mathbb{Z}[\xi]$.

For the group algebra $\mathbb{Z}W$, we have that primitive minimal positive elements of the centre $Z(\mathbb{Z}W)$ are conjugacy class sums, and so form a $\mathbb{Z}$-basis of the centre. The analogous result would be that $Z(\mathcal{H})^+_{\min}$ is a $\mathbb{Z}[\xi]$-basis for $Z(\mathcal{H})$, and this is our main result:

(2.2) MAIN THEOREM. *Let W be any Weyl group. Then*

- (i) $Z(\mathcal{H})^+_{\min}$ *is a* $\mathbb{Z}[\xi]$*-basis for* $Z(\mathcal{H})$,
- (ii) $h \in Z(\mathcal{H})^+_{\min}$ *if and only if*
  - (a) $h|_{\xi=0} = \tilde{T}_C$ *for some* $C \in \mathrm{ccl}(W)$, *and*
  - (b) $h - \tilde{T}_C$ *contains no shortest elements of any conjugacy class.*

The foundations for the proof of this are laid down in Section 4, where the result is reduced to the existence of elements h with similar characteristics as shown in (2.2)(ii) above.

We will first present the case when $\mathcal{H}$ is type A_n (in Section 5), as these results were achieved independently of the result of Geck and Rouquier. Theorem 2.2 for the type A_n case appears in (5.4). The type A_n results also use only elementary methods, and do not require any character theory, so may be interesting in their own right as an extension of the work of Jones. The general Weyl group case is presented in Section 6, where we need the existence of the elements described by Geck and Rouquier (using character theory) to get our results. The proof of (2.2) for general Weyl groups is given after (6.1).

3. THE CENTRALIZER OF AN ELEMENT ASSOCIATED TO A SIMPLE REFLECTION IN $\mathcal{H}$

We will call the set of elements conjugate under $s \in S$ an s-conjugacy class. (This is sometimes called the orbit of $\langle s\rangle$ in W, under the conjugation action.) Every s-conjugacy class is contained in a double coset $\langle s\rangle d\langle s\rangle$, for some $d \in \mathfrak{D}_{\langle s\rangle, \langle s\rangle}$, the set of distinguished $\langle s\rangle$-$\langle s\rangle$ double coset representatives in W. The double cosets may be classified according to whether the intersection $\langle s\rangle^d \cap \langle s\rangle = 1$ or $\langle s\rangle$, and this provides a means of listing all the possible types of s-conjugacy class.

If the intersection is 1, then $ds \neq sd$, and otherwise $ds = sd$. So every double coset either consists of elements $\{d, ds, sd, sds\}$ in the trivial intersection case, or $\{d, ds\}$ in the non-trivial intersection case. We can then list the s-conjugacy classes as follows: if $ds \neq sd$, we have $\{d, sds\}$ and $\{sd, ds\}$; if $ds = sd$ we have $\{d\}$ and $\{ds\}$.

The basis for the centralizer of s in $\mathbb{Z}W$ then is the set of all elements of form d or ds if $ds = sd$, and $d + sds$ or $ds + sd$ if $ds \neq sd$, and these are the minimal elements of $(Z_{\mathbb{Z}W}(s), \leq_0)$. We will give the Iwahori–Hecke algebra analogy in (3.9).

We define the centralizer of the subalgebra generated by $\tilde{T}_s$ in $\mathscr{H}$ to be the set

$$Z_{\mathscr{H}}(\tilde{T}_s) := \{h \in \mathscr{H}\colon h\tilde{T}_s = \tilde{T}_s h\}.$$

The following lemma uses methods borrowed from those in [DJ, (2.4)].

(3.1) LEMMA. *Let $c = \sum_{w\in W} r_w \tilde{T}_w$ for $r_w \in \mathbb{Z}[\xi]$. Then c is in $Z_{\mathscr{H}}(\tilde{T}_s)$ if and only if for all d distinguished in $\langle s\rangle d\langle s\rangle$ such that $sd \neq ds$ we have*

(i) $r_{ds} = r_{sd}$, *and*

(ii) $r_{sds} = r_d + \xi r_{ds}$.

Proof. First c is in the centralizer if and only if the sum of terms from each double coset commutes with $\tilde{T}_s$.

Given any double coset with d distinguished, if $ds = sd$ then the double coset consists of the elements d and ds, and each corresponding element $\tilde{T}_d$ and $\tilde{T}_{ds}$ commutes with $\tilde{T}_s$. Thus the sum $r_d\tilde{T}_d + r_{ds}\tilde{T}_{ds}$ commutes with $\tilde{T}_s$ for any $r_d, r_{ds} \in \mathbb{Z}[\xi]$.

If $ds \neq sd$, then the double coset sum is $r_d\tilde{T}_d + r_{ds}\tilde{T}_{ds} + r_{sd}\tilde{T}_{sd} + r_{sds}\tilde{T}_{sds}$. This commutes with $\tilde{T}_s$ if and only if

$$\tilde{T}_s(r_d\tilde{T}_d + r_{ds}\tilde{T}_{ds} + r_{sd}\tilde{T}_{sd} + r_{sds}\tilde{T}_{sds}) = (r_d\tilde{T}_d + r_{ds}\tilde{T}_{ds} + r_{sd}\tilde{T}_{sd} + r_{sds}\tilde{T}_{sds})\tilde{T}_s.$$

The left-hand side is

$$\begin{aligned} &r_d\tilde{T}_{sd} + r_{ds}\tilde{T}_{sds} + r_{sd}(\tilde{T}_d + \xi\tilde{T}_{sd}) + r_{sds}(\tilde{T}_{ds} + \xi\tilde{T}_{sds}) \\ &\quad = r_{sd}\tilde{T}_d + r_{sds}\tilde{T}_{ds} + (r_d + \xi r_{sd})\tilde{T}_{sd} + (r_{ds} + \xi r_{sds})\tilde{T}_{sds}, \end{aligned}$$

and the right-hand side is

$$\begin{aligned} &r_d\tilde{T}_{ds} + r_{ds}(\tilde{T}_d + \xi\tilde{T}_{ds}) + r_{sd}\tilde{T}_{sds} + r_{sds}(\tilde{T}_{sd} + \xi\tilde{T}_{sds}) \\ &\quad = r_{ds}\tilde{T}_d + (r_d + \xi r_{ds})\tilde{T}_{ds} + r_{sds}\tilde{T}_{sd} + (r_{sd} + \xi r_{sds})\tilde{T}_{sds}. \end{aligned}$$

Equating coefficients gives the result. ∎

This lemma has some direct and useful consequences for elements of the centre, and in particular the positive part of the centre.

(3.2) COROLLARY. If $h \in Z(\mathscr{H})$, then the coefficients of $\tilde{T}_w$ and $\tilde{T}_{C^w}$ in h are equal.

If in addition $h \in Z(\mathscr{H})^+$, then for $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}$, and $r \in \mathbb{N}[\xi]$ we have:

(i) $r\tilde{T}_d \leq h \Longrightarrow r(\tilde{T}_d + \tilde{T}_{sds}) \leq h$,

(ii) $r\tilde{T}_{ds}$ or $r\tilde{T}_{sd} \leq h \Longrightarrow r(\tilde{T}_{ds} + \tilde{T}_{sd} + \xi\tilde{T}_{sds}) \leq h$,

(iii) $\tilde{T}_{sds} \leq h \Longrightarrow \tilde{T}_d + \tilde{T}_{sds} \leq h$.

(3.3) PROPOSITION. *Suppose $w \in W$ is not minimal in its conjugacy class, and $c \in Z(\mathscr{H})$. Then the coefficient of $\tilde{T}_w$ in c is an $\mathbb{N}[\xi]$-linear combination of coefficients of strictly shorter elements in c. In fact, it is an $\mathbb{N}[\xi]$-linear combination of the coefficients of shortest elements of conjugacy classes.*

Proof. The first statement follows from the first statement of (3.2), (1.2), and (3.1)(ii). The second follows by induction. ∎

(3.4) COROLLARY. *If there exists an $h \in Z(\mathscr{H})$, such that $h_0 = a\tilde{T}_C$ for $a \in \mathbb{Z}$ and there are no shortest elements from any conjugacy class in $h - h_0$, then h is unique with these properties.*

Proof. Suppose $h' \in Z(\mathscr{H})$ has the property that $h'_0 = a\tilde{T}_C$ and $h' - h'_0$ has no shortest elements from any conjugacy class. Then $h' - h \in Z(\mathscr{H})$ has no shortest elements of any conjugacy class. Thus by (3.3), $h' - h = 0$. ∎

We return to the centralizer of $\tilde{T}_s$ in the Iwahori–Hecke algebra. The minimal $\mathbb{Z}$-basis for $Z_{\mathbb{Z}W}(s)$ is the set of s-conjugacy class sums, as noted at the start of this section. We now provide the analogy in $\mathscr{H}$.

(3.5) DEFINITION. Let d be distinguished in $\langle s\rangle d\langle s\rangle$. We define the following four types of elements, and call them *s-class elements* because they correspond to s-conjugacy classes:

Type I, $d \in Z_W(s)$:	$b^{\mathrm{I}}_d = \tilde{T}_d$,
	$b^{\mathrm{I}}_{ds} = \tilde{T}_{ds}$,
Type II, $d \notin Z_W(s)$:	$b^{\mathrm{II}}_d = \tilde{T}_d + \tilde{T}_{sds}$,
	$b^{\mathrm{II}}_{ds} = \tilde{T}_{ds} + \tilde{T}_{sd} + \xi\tilde{T}_{sds}$.

Note that every distinguished $\langle s\rangle$-$\langle s\rangle$-double coset representative either commutes with s or it does not. Also note that when $\xi = 0$, these elements correspond to sums of s-conjugate elements, and each element of the $\langle s\rangle$-$\langle s\rangle$-double coset appears with coefficient 1 in exactly one s-class element.

Later we will use diagrams to represent the structure of central elements, and the core "cells" of these diagrams will be those corresponding to s-class elements. The Type II s-class elements may be represented graphically by the diagrams

$$
\begin{tikzcd}
\tilde{T}_d \arrow[d, "s"'] & \tilde{T}_{ds} \arrow[rr, dash, "s"] \arrow[dr, "s"] & & \tilde{T}_{sd} \arrow[dl, "s"'] \\
\tilde{T}_{sds} & & \xi\tilde{T}_{sds} & \\
b^{\mathrm{II}}_d & & b^{\mathrm{II}}_{ds} &
\end{tikzcd}
\tag{3.6}
$$

(3.7) Proposition. *The set of s-class elements* $\{b^{\mathrm{I}}_d, b^{\mathrm{I}}_{ds}, b^{\mathrm{II}}_d, b^{\mathrm{II}}_{ds} \mid d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}\}$ *is a* $\mathbb{Z}[\xi]$*-basis for* $Z_{\mathscr{H}}(\tilde{T}_s)$.

Proof. On specialization to $\xi = 0$, each s-class element becomes a sum of s-conjugate elements in the group algebra. Such sums are a basis for the centralizer of s in the group algebra, and in particular are linearly independent. It follows that the s-class elements are also linearly independent. Their centrality is also easy to check.

Let h be an element of $Z_{\mathscr{H}}(\tilde{T}_s)$, and write r_w for the coefficient of $\tilde{T}_w$ in h. Then, as in the proof for 3.1, we may write h as a $\mathbb{Z}[\xi]$-linear combination of sums of terms corresponding to elements in an $\langle s\rangle$-$\langle s\rangle$ double coset. If the distinguished representative of the double coset is in the centralizer $Z_W(s)$, then $r_d\tilde{T}_d + r_{ds}\tilde{T}_{ds} = r_d b^{\mathrm{I}}_d + r_{ds} b^{\mathrm{I}}_{ds}$—a linear combination of s-class elements—so we need only to check the case when $d \notin Z_W(s)$. Using the relations from (3.1), we have

$$
\begin{aligned}
r_d\tilde{T}_d + r_{ds}\tilde{T}_{ds} + r_{sd}\tilde{T}_{sd} + r_{sds}\tilde{T}_{sds} &= r_d\tilde{T}_d + r_{ds}\tilde{T}_{ds} + r_{ds}\tilde{T}_{sd} + (r_d + \xi r_{ds})\tilde{T}_{sds} \\
&= r_d(\tilde{T}_d + \tilde{T}_{sds}) + r_{ds}(\tilde{T}_{ds} + \tilde{T}_{sd} + \xi\tilde{T}_{sds}),
\end{aligned}
$$

which is a linear combination of s-class elements. Thus we have that any $h \in Z_{\mathscr{H}}(\tilde{T}_s)$ may be written

$$
h = \sum_{\substack{d\in\mathfrak{D}_{\langle s\rangle,\langle s\rangle}\\ d\in Z_W(s)}} \left(r_d b^{\mathrm{I}}_d + r_{ds} b^{\mathrm{I}}_{ds}\right) + \sum_{\substack{d\in\mathfrak{D}_{\langle s\rangle,\langle s\rangle}\\ d\notin Z_W(s)}} \left(r_d b^{\mathrm{II}}_d + r_{ds} b^{\mathrm{II}}_{ds}\right),
$$

where r_w is the coefficient of $\tilde{T}_w$ in h. Thus h is a linear combination of s-class elements, and spanning follows. ∎

(3.8) Corollary. *Let $\mathfrak{D}_{\langle s\rangle,\langle s\rangle}$ be the set of distinguished $\langle s\rangle$-$\langle s\rangle$-double coset representatives in W. Then* $\dim Z_{\mathscr{H}}(\tilde{T}_s) = 2|\mathfrak{D}_{\langle s\rangle,\langle s\rangle}|$.

Proof. For each double coset, there are two basis elements for $Z_{\mathscr{H}}(\tilde{T}_s)$. ∎

An element of the centre is also an element of any centralizer in $\mathcal{H}$, and in particular the centralizers of elements $\tilde{T}_s$ for $s \in S$. Using the same principle as in Corollary 3.4, it may thus be represented as a graph with terms of the form $\xi^k \tilde{T}_w$ as nodes, and lines labelled by simple reflections connecting each node with the other terms in its s-class element for each $s \in S$. This provides a graphical way to check if an element is in the centre: ensure that for each $s \in S$ every node is part of a unique s-class element subgraph.

(3.9) LEMMA. *The set of all s-class elements is the set* $Z_{\mathcal{H}}(\tilde{T}_s)^+_{\min}$.

Proof. The s-class elements are all clearly primitive, so we need to show they are minimal in $Z_{\mathcal{H}}(\tilde{T}_s)^+$.

Every positive element of the centralizer can be written as an $\mathbb{N}[\xi]$-multiple of an element which has non-zero specialization (to $\xi = 0$), so it suffices to show that any element with non-zero specialization is greater than an s-class element. Let h be such an element.

The fact that h has non-zero specialization means that there is at least one term $r_w \tilde{T}_w \leq h$ with integer coefficient $r_w \in \mathbb{N}$, and w must belong to some $\langle s \rangle$-$\langle s \rangle$ double coset with distinguished representative d say. We may suppose that $ds \neq sd$ since $ds = sd$ implies $r_w \tilde{T}_w$ would be a $\mathbb{N}$-multiple of an s-class element of Type I on its own, meaning that an s-class element would certainly be less than h.

Note that since r_w is an integer, any integer multiple of $\tilde{T}_w$ will be less than h so long as the integer is less than r_w. In particular, $\tilde{T}_w \leq h$. Thus for simplicity we may suppose that at least one of $\tilde{T}_d$, $\tilde{T}_{sd}$, $\tilde{T}_{ds}$, or $\tilde{T}_{sds}$ must be less than h, for some d. Then (3.2) gives that h is greater than some s-class element.

This completes the proof. ∎

4. CONSTRUCTING CENTRAL ELEMENTS

Given the s-class element basis for the centralizer of $\tilde{T}_s$ in $\mathcal{H}$ for any s, we may write a central element as a linear combination of s-class elements for any $s \in S$. Indeed, for any element $h \in \mathcal{H}^+$ and $s \in S$ we may write h as a linear combination h_s of s-class elements, plus a linear combination h'_s of terms $\xi^k \tilde{T}_w$ which are neither $\mathbb{N}[\xi]$-multiples of complete s-class elements (of type I) on their own, nor can be summed with any other terms in h'_s to create an $\mathbb{N}[\xi]$-multiple of an s-class element. In other words, h_s is a maximal linear combination of s-class elements less than or equal to h.

[Note however that we do not claim that h_s is the unique maximal linear combination of s-class elements less than h. This is not possible in general, as for example we could have $\xi^k \tilde{T}_{sds} + \xi^k \tilde{T}_d + \xi^{k-1} \tilde{T}_{sd} + \xi^{k-1} \tilde{T}_{ds}$ less than

h for some $d \in \mathfrak{D}_{\langle s \rangle, \langle s \rangle}$, and then either $\xi^k(\tilde{T}_d + \tilde{T}_{sds})$ or $\xi^{k-1}(\tilde{T}_{ds} + \tilde{T}_{sd} + \xi\tilde{T}_{sds})$ are s-class elements which could be put into h_s.]

Having decomposed $h \in \mathcal{H}^+$ in such a manner into $h = h_s + h'_s$, we may then add terms to complete s-class elements containing the terms in h'_s. That is, we may add terms to h to create a new element (say $\bar{h} \geq h$) which is a linear combination of s-class elements. In other words, $\bar{h}$ is a minimal element of $Z_{\mathcal{H}}(\tilde{T}_s)$ greater than or equal to h. (Note again we do not claim uniqueness for such a minimal centralizer element greater than h. For all terms in h'_s not of the form $\xi^k\tilde{T}_{sds}$ (for $d \in \mathfrak{D}_{\langle s \rangle, \langle s \rangle}$) the added terms will in fact be unique, but the case of $\xi^k\tilde{T}_{sds}$ could be considered either as part of the s-class element b_d^{II} or b_{ds}^{II}, if $k \geq 1$).

This describes the nucleus of an algorithm for constructing a central element containing a given non-central element. We could continue to construct centralizer elements corresponding to different $s \in S$ by adding more and more terms until we (hopefully) eventually create an element in all centralizers—the centre. To ensure the algorithm stops, however (and does not continue to add elements ad infinitum), we need to either: ensure that terms of the form $\xi^k\tilde{T}_{sds}$ for $d \in \mathfrak{D}_{\langle s \rangle, \langle s \rangle}$ never occur in h'_s for any s and for any stage in the process (or define the algorithm more closely to ensure this); or provide an upper bound in the centre which controls the additions to h.

Assuming that for any $h \in \mathcal{H}^+$ we can find a positive central element c greater than h (which we can certainly do: for example, $h \leq N_{W,1}(h) \in Z(\mathcal{H})$—see (5.1) for a definition), we can apply the latter approach to ensure we can always construct a central element greater than or equal to h and less than or equal to c, no matter in what order of additions we proceed. If we need to complete an s-class element for which $\xi^k\tilde{T}_{sds}$ is in h, we need to choose whether to consider it part of b_d^{II} or b_{ds}^{II}—in other words, whether to add $\xi^k\tilde{T}_d$ or $\xi^{k-1}(\tilde{T}_{sd} + \tilde{T}_{ds})$. We can decide this on the basis of which is less than $c - h$. Then our new element will remain less than or equal to c. If both options are less than $c - h$, the choice can be arbitrary. Terms other than $\xi^k\tilde{T}_{sds}$ are in uniquely defined s-class elements, and so a choice will never need to be made (see Lemma 3.2(ii) and (iii)).

Suppose $h = h_s + h'_s \in \mathcal{H}^+$, with h_s a maximal element of $Z_{\mathcal{H}}(\tilde{T}_s)$ less than h. Let m_s be the length of the shortest term in h'_s (for non-zero h'_s). We now formalize the above, adding some extra conditions, with the following definition:

(4.1) DEFINITION. Let $h \in \mathcal{H}^+$, with $h \leq c \in Z(\mathcal{H})^+$. Define the algorithm $\mathfrak{B}_c$ to conduct the following sequence of procedures:

(i) split h into $h = h_s + h'_s$ for $s \in S$ with h_s maximal in $Z_{\mathcal{H}}(\tilde{T}_s)$ less than or equal to h;

(ii) if $h'_s = 0$ for all $s \in S$, stop;

(ii′) otherwise, evaluate m_s for each s, and choose $s \in S$ such that m_s is minimal;

(iii) add terms from $c - h$ which complete the s-class elements containing terms in h'_s of length m_s.

(iv) declare the new element to be $\mathfrak{B}_c(h)$, and repeat from (i) with a new element.

Note that here we do not make h into an element of a centralizer of $\tilde{T}_s$ for some s immediately. We find the shortest term in h which is not in a complete s-class element for some $s \in S$, and add the necessary terms to make that particular s-class element complete. The purpose of this aspect of the definition is that later we will use induction on the length m_s of the shortest term in an incomplete s-class element.

An immediate consequence of the definition is the following.

(4.2) LEMMA. *Suppose that after n iterations of $\mathfrak{B}_c$ the shortest term of $\mathfrak{B}_c^n(h)$ in an incomplete s-class element for some $s \in S$ has length k—that is, $m_s = k$ in $\mathfrak{B}_c^n(h)$. Then every element of length $< k$ in $\mathfrak{B}_c^n(h)$ is in a complete s-class element, for all $s \in S$.*

Our main aim is to construct basis elements for the centre which specialize to the conjugacy class sum $\tilde{T}_C$. We now show that given the existence of certain types of positive central elements to use as upper bounds, we can start the algorithm on $\tilde{T}_C$ and never need to add shorter elements. With this, the algorithm becomes uniquely defined, and it is never necessary to make a choice relative to an upper bound. In other words, we will show paradoxically that given the existence of certain types of element which we can use as an upper bound, upper bounds are not necessary when starting with $\tilde{T}_C$.

(4.3) PROPOSITION. *Suppose that for each conjugacy class C there exists an element $L_C \in Z(\mathscr{H})^+$ with the following properties*:

(L1) $L_C|_{\xi=0} = a\tilde{T}_C$ *for some* $a \in \mathbb{N}$; *and*

(L2) $L_C - a\tilde{T}_C$ *contains no shortest elements from any conjugacy class.*

Then $\mathfrak{B}^i_{L_C}(a\tilde{T}_C)$ never needs to refer to the upper bound L_C for any $i \in \mathbb{N}$.

Proof. There is only need to refer to the upper bound if it is necessary at some point to decide whether to consider an element of the form $\xi^k\tilde{T}_{sds}$ (for some $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}$) as part of the s-class element b^{II}_d or b^{II}_{ds}. Equivalently, we will need to refer to the upper bound if it is necessary at some point to add a shorter element or elements. We claim that under the conditions of the proposition it is never necessary to add shorter elements at any point in the construction, and we prove this by induction on the number of repeats of $\mathfrak{B}_{L_C}$.

Consider the first additions made to $a\tilde{T}_C$ via $\mathfrak{B}_{L_C}$. Since for any $s \in S$, $a\tilde{T}_C$ may be written as a linear combination of sums of s-conjugate elements, the only s-class elements which could possibly be incomplete are those of type $b^{\mathrm{II}}_{ds} = \tilde{T}_{ds} + \tilde{T}_{sd} + \xi\tilde{T}_{sds}$ for some $d \in \mathfrak{D}_{\langle s\rangle, \langle s\rangle}$. Thus the only additions to $a\tilde{T}_C$ will be of form $a\xi\tilde{T}_{sds}$ for some $s \in S$ and $d \in \mathfrak{D}_{\langle s\rangle, \langle s\rangle}$, and these are uniquely determined without reference to L_C.

Suppose by induction that after k repeats of $\mathfrak{B}_{L_C}$, no shorter additions have been made—equivalently, no choice has been made at any point in the construction so far: no reference to L_C has been required.

Then all terms of length shorter than m_s in $\mathfrak{B}^k_{L_C}(a\tilde{T}_C)$ are in complete s-class elements for all $s \in S$ (as pointed out in Lemma 4.2).

Now suppose a shorter addition were required to complete the s-class element containing $\xi^i\tilde{T}_w$ of length $l(w) = m_s$ in $\mathfrak{B}^k_{L_C}(a\tilde{T}_C)$. Then clearly $w = sds$ for some $d \in \mathfrak{D}_{\langle s\rangle, \langle s\rangle}$, and we will need to add either $\xi^i\tilde{T}_d$ or $\xi^{i-1}(\tilde{T}_{ds} + \tilde{T}_{sd})$. The added element, being strictly shorter, will also reduce m_s for $\mathfrak{B}^{k+1}_{L_C}(a\tilde{T}_C)$, since there were no incomplete elements of that length or shorter in $\mathfrak{B}^k_{L_C}(a\tilde{T}_C)$.

We will then need to add all of $\xi^i\tilde{T}_{C^d}$ (resp. $\xi^{i-1}\tilde{T}_{C^{ds}}$) if $\xi^i\tilde{T}_d$ (resp. $\xi^{i-1}(\tilde{T}_{ds} + \tilde{T}_{ds})$) is added, by (4.1), without increasing m_s. Thus we will be adding at least one further element which cancels with some $s \in S$, and which will thus require further strictly shorter additions via the algorithm (by Corollary 1.2). This will continue, so long as $d \notin C_{\min}$ (resp. $ds \notin C_{\min}$). Thus, in a finite number of steps (since all of C^d (resp. C^{ds}) will be added in a finite number of steps), we will add shortest elements of some conjugacy class.

But these additions must come from $h = L_C - \mathfrak{B}^{k'}_{L_C}(a\tilde{T}_C)$ (where the k'th step is the one requiring the addition of shorter elements), and since there are no shortest elements from any conjugacy class in h (the only shortest elements in L_C are those in $a\tilde{T}_C$, which are also in $\mathfrak{B}^i_{L_C}(a\tilde{T}_C)$ for any $i \in \mathbb{N}$), we have a contradiction. Thus there is never a need to add strictly shorter elements at any point in the algorithm, and hence we never need to refer to the upper bound L_C. ∎

Thus we have that under specific conditions (that we start with $a\tilde{T}_C$ and that elements with the properties of L_C exist), the algorithm is well defined without reference to any upper bound at all. This motivates us to make the following definition of a simpler algorithm.

(4.4) DEFINITION. Let $h \in \mathscr{H}^+$. Define the algorithm $\mathfrak{A}$ to conduct the following procedures.

(i) Split h into $h = h_s + h'_s$ for each $s \in S$ such that h_s is maximal in $Z_{\mathscr{H}}(\tilde{T}_s)$ less than or equal to h;

(ii) if $h'_s = 0$ for all $s \in S$, stop;

(ii)′ otherwise evaluate m_s for each s such that $h'_s \neq 0$, and choose $s \in S$ such that m_s is minimal;

(iii) add terms to h which complete the s-class elements of those terms in h'_s of length m_s;

(iv) declare the new element to be $\mathfrak{A}(h)$, and repeat from (i) with the new element.

(4.5) THEOREM. *The following are equivalent*:

(i) *There exists an element* $L_C \in Z(\mathscr{H})^+$ *which satisfies* (L1) *and* (L2) *from* (4.3);

(ii) *The algorithm* $\mathfrak{A}^i(\tilde{T}_C)$ *is well defined for all* $i \in \mathbb{N}$;

(iii) *There exists an element* $\Gamma_C = \mathfrak{A}^n(\tilde{T}_C) \in Z(\mathscr{H})^+$ *for some* $n \in \mathbb{N}$ *which satisfies*

(G1) $\Gamma_C|_{\xi=0} = \tilde{T}_C$, *and*

(G2) $\Gamma_C - \tilde{T}_C$ *contains no shortest elements of any conjugacy class.*

Proof. Part (iii) clearly implies (i), since (L1) and (L2) are satisfied by Γ_C with $a = 1$, so it will suffice to show (i) implies (ii) and (ii) implies (iii).

We have shown (in (4.3)) that when started with $a\tilde{T}_C$, the algorithm $\mathfrak{B}_{L_C}$ does not refer to the upper bound L_C. Thus $\mathfrak{B}^i_{L_C}(a\tilde{T}_C) = \mathfrak{A}^i(a\tilde{T}_C)$ for all $i \in \mathbb{N}$, and there is an integer n such that $\mathfrak{A}^n(a\tilde{T}_C) \in Z(\mathscr{H})$. Now since all additions via $\mathfrak{A}$ starting with $a\tilde{T}_C$ are well defined (same length or longer), they all carry the factor of a. That is, all new additions at the ith step either have the same coefficient as a term in $\mathfrak{A}^{i-1}(a\tilde{T}_C)$, or they have a coefficient which is a ξ-multiple of a coefficient in $\mathfrak{A}^{i-1}(a\tilde{T}_C)$. Thus every term added via $\mathfrak{A}$ to $a\tilde{T}_C$ is a $\mathbb{N}[\xi]$-multiple of a.

Thus $\frac{1}{a}\mathfrak{A}^n(a\tilde{T}_C)$ is in $Z(\mathscr{H})^+$ for some $n \in \mathbb{N}$. That is, all coefficients in $\frac{1}{a}\mathfrak{A}^n(a\tilde{T}_C)$ are from $\mathbb{N}[\xi]$. Further, $\frac{1}{a}\mathfrak{A}^n(a\tilde{T}_C)|_{\xi=0} = \tilde{T}_C$, and so it satisfies (L1).

Now $\mathfrak{A}^n(a\tilde{T}_C)$ contains no shortest elements of non-zero coefficient except those in $a\tilde{T}_C$, since all initial additions to incomplete s-class elements in $a\tilde{T}_C$ were of form $\xi\tilde{T}_{sds}$ for some $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}$, $s \in S$ (in particular, not shortest in their conjugacy class), and all ensuing additions have been same length or longer (and thus also not shortest in their conjugacy classes). Thus $\frac{1}{a}\mathfrak{A}^n(a\tilde{T}_C)$ also satisfies (L2). Hence we have the conditions to apply (4.3) with respect to $L_C = \frac{1}{a}\mathfrak{A}^n(a\tilde{T}_C)$, starting with $\tilde{T}_C$. Part (ii) follows.

To get part (iii) from (ii), we simply need to apply $\mathfrak{A}$ to $\tilde{T}_C$ sufficiently many times to get a central element. Since $\mathfrak{A}$ is well defined starting from

$\tilde{T}_C$, it will only add same length or longer, and thus in a finite number of steps the shortest element not in a complete s-class element (m_s) will increase (for any $r\tilde{T}_w$ with $l(w) = m_s$, only same-length additions will be the set C^w, which is finite. Once all terms of C^w are added, the only possible additions will be strictly longer). Of course, there can be only finitely many increases in m_s, since W is a finite group, and so there exists an $n \in \mathbb{N}$ such that $\mathfrak{A}^n(\tilde{T}_C) \in Z(\mathscr{H})^+$. By the same arguments as used showing (i) implies (ii), we also have that $\mathfrak{A}^n(\tilde{T}_C)|_{\xi=0} = \tilde{T}_C$, and there are no shortest elements of any conjugacy class in $\mathfrak{A}^n(\tilde{T}_C)$ other than those from $\tilde{T}_C$. Thus (G1) and (G2) are satisfied, and this completes the proof. ∎

(4.5.1) *Remark.* It is sufficient to start $\mathfrak{A}$ on the sum of the minimal elements of C, since by (1.1) every element of C can be obtained from a shortest element by a non-decreasing series of conjugations by simple reflections. Examples of the algorithm for types B_2 and A_3 are given in Sections 8.2 and 8.4, respectively.

(4.6) Corollary. *Let* $C \in \mathrm{ccl}(W)$, *and suppose there exists a* Γ_C *satisfying* (G1) *and* (G2) *from* (4.5). *If* $w_C \in C_{\min}$ *and* $r\tilde{T}_{w_C} \le h \in Z(\mathscr{H})^+$, *then* $r\Gamma_C \le h$.

Proof. First, if $r\tilde{T}_{w_C} \le h \in Z(\mathscr{H})^+$, then all elements of $C_{\min}$ appear in h with the same coefficient r. (This was shown independently by both Starkey [C2] and Ram [R] in type A_n by showing the irreducible characters take the same value on all shortest elements of a conjugacy class. The result was extended to all types of Weyl group by Geck and Pfeiffer in [GP]. The type A_n result can also be obtained by viewing $W(A_n)$ as the symmetric group on $n+1$ letters. Each conjugacy class corresponds to a partition of $n+1$ which gives a cycle type for the permutation. One can show that shortest elements corresponding to the same composition of $n+1$ have the same coefficient by showing they are all in the same equivalence class C^w (see (3.2)). Then a lemma of Dipper and James (in [DJ, (2.11)]) shows the coefficients of shortest elements corresponding to different compositions but the same partition are the same.) Then $r\tilde{T}_C \le h$ also, by Remark 4.5.1. The remainder follows by noticing that the same-length and longer additions via the algorithm $\mathfrak{A}$ are in fact implications, by (3.2)(ii) and (iii). Thus since $\mathfrak{A}$ is well defined when started on $\tilde{T}_C$ (and so we add only same length or longer starting with $\tilde{T}_C$), we have a chain of necessity which gives the whole of Γ_C less than h. ∎

(4.7) Theorem. *If there exists an element* $\Gamma_C \in Z(\mathscr{H})^+$ *satisfying* (G1) *and* (G2) *from* (4.5), *then*

(i) Γ_C *is the unique element of* $Z(\mathscr{H})^+$ *satisfying* (G1) *and* (G2).

If for each conjugacy class C *there exists an element* $\Gamma_C \in Z(\mathscr{H})^+$ *satisfying* (G1) *and* (G2), *then*

(ii) *The set* $\{\Gamma_C \mid C \in \mathrm{ccl}(W)\}$ *is a* $\mathbb{Z}[\xi]$*-basis for* $Z(\mathscr{H})$, *and*

(iii) $Z(\mathscr{H})^+_{\min} = \{\Gamma_C \mid C \in \mathrm{ccl}(W)\}$.

Proof. (i) Suppose there exists a $\Gamma'_C \in Z(\mathscr{H})^+$ satisfying (G1) and (G2). Then $\Gamma_C - \Gamma'_C$ has no shortest elements of any conjugacy class with non-zero coefficient. This contradicts (4.2) unless $\Gamma_C = \Gamma'_C$.

(ii) Linear independence follows by specialization (putting $\xi = 0$ gives a set of conjugacy class sums, which are a basis for the centre of the group algebra, and so linearly independent).

For spanning, we can use the fact that $\mathscr{H}_{\mathbb{Q}(\xi)}$ is isomorphic to $\mathbb{Q}(\xi)W$ (see, for example, [CR, Sect. 68] or [C1, Sect. 10.11]), which gives that the set is a basis for the centre over $\mathbb{Q}(\xi)$ (because it is a linearly independent set of the same rank). Any element of $Z(\mathscr{H})$ may then be written as a $\mathbb{Q}(\xi)$-linear combination of the Γ_C, since $Z(\mathscr{H}) \subseteq Z(\mathscr{H}_{\mathbb{Q}(\xi)})$. Let $h \in Z(\mathscr{H})$ and write $h = \sum_C r_C \Gamma_C$, for $r_C \in \mathbb{Q}(\xi)$.

Since $h \in Z(\mathscr{H})$, the coefficients of the shortest elements of a conjugacy class C in h are from $\mathbb{Z}[\xi]$, and yet the only occurrences of the shortest elements of C in h are from Γ_C, and these appear with coefficient r_C in h—since the coefficient of the shortest elements of C in Γ_C is 1. Thus we have that $r_C \in \mathbb{Z}[\xi]$ for all conjugacy classes C, and h is in the $\mathbb{Z}[\xi]$-span of the set of Γ_C's.

(iii) We need to show first that each Γ_C is in fact minimal, and then that there are no other minimal elements.

Take any Γ_C, and suppose there were another non-zero positive central element $h \leq \Gamma_C$. Since h is non-zero and central, it must contain shortest elements of some conjugacy class with non-zero coefficients (by (3.3)). But $h \leq \Gamma_C$, so h must contain shortest elements from C, with coefficient less than the coefficient of the same shortest elements in Γ_C. By (4.6), we then must have $\Gamma_C \leq h$ and so $\Gamma_C = h$.

Suppose there was a minimal element h of $Z(\mathscr{H})^+$ which is not equal to Γ_C for any C. By (ii), we may write h as a $\mathbb{Z}[\xi]$-linear combination of the Γ_C's, $h = \sum_C r_C \Gamma_C$, for $r_C \in \mathbb{Z}[\xi]$. Since h is positive, the coefficients of any shortest elements in h are positive. Since shortest elements of a conjugacy class C occur in only one Γ_C this means that the coefficients r_C of Γ_C in the expansion of h must be positive, for each C. That is, h is an $\mathbb{N}[\xi]$-linear combination of the Γ_C's, which then means that any Γ_C with non-zero coefficient in the expression of h is *less than* h, contradicting the minimality of h, unless $h = \Gamma_C$ for some C. ∎

(4.8) COROLLARY. *Suppose there exist elements* L_C *satisfying* (L1) *and* (L2) *from* (4.3)*. Then* $Z(\mathscr{H})^+_{\min}$ *is a* $\mathbb{Z}[\xi]$*-basis for* $Z(\mathscr{H})$.

Proof. By (4.5), the existence of elements L_C satisfying (L1) and (L2) is equivalent to the existence of elements Γ_C satisfying (G1) and (G2) (see (4.5)). Then (4.7)(ii) and (iii) give the desired conclusion. ∎

Reverting To a Basis over $\mathbb{Z}[q, q^{-1}]$

We now demonstrate how to obtain the analogous basis for the centre over $\mathbb{Z}[q, q^{-1}]$. Of course one can immediately obtain corresponding central elements over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ by substituting $\xi = q^{1/2} - q^{-1/2}$ and $\tilde{T}_w = q^{-(l(w))/2} T_w$ in Γ_C for each C. We will show that these are either over $\mathbb{Z}[q, q^{-1}]$ or are $q^{-1/2}$-multiples of elements over $\mathbb{Z}[q, q^{-1}]$. We begin with a result on the coefficients of terms in Γ_C:

(4.9) LEMMA. *Suppose Γ_C exists and $\xi^i \tilde{T}_w \leq \Gamma_C$. Then*

(i) *i is even if and only if $l(w) = l_C + 2k$ for some $k \geq 0$, and*

(ii) *i is odd if and only if $l(w) = l_C + 2k + 1$ for some $k \geq 0$.*

Furthermore, we always have $i \leq l(w) - l_C$.

Proof. Consider how additions of terms of different lengths and different coefficients may arise due to the algorithm. The only way the power of ξ is increased is by adding $\xi r \tilde{T}_{sds}$ to complete the s-class element $r b^{\mathrm{II}}_{ds}$ (for $r \in \mathbb{Z}[\xi]$), and this addition is also the only way the length of the element can be increased by an odd number—1. All other completions (for s-class elements b^{II}_d, or of form $\tilde{T}_{ds}$ in b^{II}_{ds}) maintain the power of ξ and add terms of length 2 greater than that already present.

Thus, even (resp. odd) powers of ξ and terms of length an even (resp. odd) difference from l_C arise only by an even (resp. odd) sequence of b^{II}_{ds} completions of form $\xi \tilde{T}_{sds}$ (interspersed perhaps with b^{II}_d completions).

Increases in the power of ξ are linked to an increase by one in the length of the word. Thus the maximum power of ξ possible in the coefficient of $\tilde{T}_w$ would be if we were to increase the power by one for every increase by one in word length, from the shortest word in C up to the addition of $\xi^i \tilde{T}_w$. This gives a maximum increase in power of ξ (from a power of zero—a coefficient of one) of $l(w) - l_C$. ∎

If we write $\Gamma_{C,q}$ for the image of Γ_C in the injection $\mathscr{H} \to \mathscr{H}_q$ (defined by setting $\xi = q^{1/2} - q^{-1/2}$ and $\tilde{T}_w = q^{-(l(w))/2} T_w$), then we have the following consequence of the lemma:

(4.10) PROPOSITION. *Suppose Γ_C exists for a conjugacy class C. If l_C is even, then $\Gamma_{C,q} \in \mathscr{H}_{\mathbb{Z}[q,q^{-1}]}$. If l_C is odd, then $q^{-1/2}\Gamma_{C,q} \in \mathscr{H}_{\mathbb{Z}[q,q^{-1}]}$.*

Proof. Every term in Γ_C is of form $\xi^i \tilde{T}_w$ which gives
$$(q^{1/2} - q^{-1/2})^i q^{-l(w)/2} T_w$$
in $\Gamma_{C,q}$.

If l_C is even, then by (4.9) we will have either i even and $l(w)$ even, or i odd and $l(w)$ odd. In either case we have $i + l(w)$ even, and so $(q^{1/2} - q^{-1/2})^i q^{-l(w)/2} = q^{-1/2} q^{i+l(w)}(q-1) \in \mathbb{Z}[q, q^{-1}]$.

If l_C is odd, again by (4.9) we have either i even and $l(w)$ odd, or i odd and $l(w)$ even. Then in either case we have $i + l(w)$ odd, and so $q^{-1/2}(q^{1/2} - q^{-1/2})^i q^{-l(w)/2} = q^{-1/2} q^{1+i+l(w)}(q-1) \in \mathbb{Z}[q, q^{-1}]$. ∎

(4.11) COROLLARY. *If Γ_C exists for all conjugacy classes C, then the set $\{\Gamma_{C,q} \mid C \in \mathrm{ccl}(W)$, and l_C even$\} \cup \{q^{-1/2}\Gamma_{C,q} \mid C \in \mathrm{ccl}(W)$, and l_C odd$\}$ is a $\mathbb{Z}[q, q^{-1}]$ basis for $Z(\mathcal{H}_{\mathbb{Z}[q,q^{-1}]})$.*

Proof. Each Γ_C commutes with $\tilde{T}_w$ for any $w \in W$ so $\Gamma_{C,q}$ commutes with $\tilde{T}_w = q^{-(l(w))/2} T_w$, and so commutes with all T_w. These elements are also linearly independent, as they specialize (on $q = 1$) to conjugacy class sums in the group algebra. The spanning can be shown in exactly the same way as for the $\mathbb{Z}[\xi]$ case (see proof of (4.7)(ii)). ∎

5. TYPE A_n

We now show the existence in type A_n of elements Γ_C as required by Theorem 4.7, using results of Jones from [J]. Despite the new work of Geck and Rouquier [GR] described in the next section, this proof in type A_n remains interesting as it is entirely elementary and does not need character theory.

In this section, we will refer to $W = W(A_n)$ and $\mathcal{H} = \mathcal{H}(A_n)$, where $W(A_n)$ has generators $s_1, \dots, s_n$ and relations $s_i^2 = 1$ for $1 \le i \le n$, $(s_i s_j)^2 = 1$ for $|i - j| \ge 2$, and $(s_i s_{i+1})^3 = 1$ for $1 \le i < n$. The following definition is attributed in [J] to Hoefsmit and Scott:

(5.1) *Definition.* Let W' be a parabolic subgroup of W and let $\mathfrak{D}$ be the set of distinguished right coset representatives of W' in W. For $h \in \mathcal{H}$, we define the *relative norm* of h to be

$$N_{W,W'}(h) = \sum_{d \in \mathfrak{D}} \tilde{T}_{d^{-1}} h \tilde{T}_d.$$

The following lemma is vital for the Jones results (see [J, (2.13)]):

(5.2) LEMMA. *If $h \in Z_{\mathcal{H}}(\mathcal{H}(W'))$, then $N_{W,W'}(h) \in Z(\mathcal{H})$.*

A parabolic subgroup W_λ of W corresponds to a partition λ of $n+1$ in the following way. If $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n+1$, then we define

$$W_\lambda = W_{\lambda_1} \times \dots \times W_{\lambda_r},$$

where

$$W_{\lambda_i} = \langle s_{\lambda_1+\cdots+\lambda_{i-1}+1,\ldots,} s_{\lambda_1+\cdots+\lambda_i-1} \rangle.$$

[Note that there are many compositions of $n+1$ corresponding to the same partition, each reflecting a permutation of the λ_i's. These different compositions give different (conjugate) parabolic subgroups, but we will focus on the "standard" representative corresponding to the ordering of the λ_i's which is a partition.]

We denote the Iwahori–Hecke algebra corresponding to the parabolic subgroup W_λ by $\mathscr{H}_\lambda := \mathscr{H}(W_\lambda)$.

For such a component subgroup W_{λ_i} we define w_{λ_i} to be its Coxeter element $w_{\lambda_i} = s_{\lambda_1+\cdots+\lambda_{i-1}+1} \cdots s_{\lambda_1+\cdots+\lambda_i-1}$, and write $l_\lambda = \sum_{i=1}^r (\lambda_i - 1)$ for the length of the Coxeter element of W_λ. There is also a corresponding conjugacy class C_λ of W for each partition λ of $n+1$, for which the coxeter element $w_{\lambda_1} \cdots w_{\lambda_r}$ of W_λ is a shortest representative.

Then for any partition $\lambda \vdash n+1$, let

$$\eta_\lambda = \prod_{i=1}^r N_{W_{\lambda_i-1},\,1}(\tilde{T}_{w_{\lambda_i}})$$

and

$$b_\lambda = N_{W,W_\lambda}(\eta_\lambda).$$

The main result of [J] is that the set of b_λ for $\lambda \vdash n+1$ is a $\mathbb{Q}[\xi]$-basis for $Z(\mathscr{H})$. Our main interest in these elements lies in some of the properties they have in addition to being a basis. The following properties are proven in [J] at various points in the paper:

(5.3) PROPOSITION. *Let $w \in W$ and $0 \neq r \in \mathbb{N}[\xi]$. Then*

(i) $\eta_\lambda \in Z(\mathscr{H}_\lambda)$,

(ii) $b_\lambda \in Z(\mathscr{H})^+$,

(iii) *If $r\tilde{T}_w \leq b_\lambda$, then $l(w) \geq l_\lambda$,*

(iv) *If $r\tilde{T}_w \leq b_\lambda$ and $l(w) = l_\lambda$, then $w \in C_{\lambda,\min}$ and $r \in \mathbb{N}$,*

(v) *$b_\lambda|_{\xi=0} = a\tilde{T}_{C_\lambda}$ for some $a \in \mathbb{N}$.*

Proof. (i) [J, (3.23)].

(ii) The centrality of b_λ follows from (5.2) and (i), and the positivity follows since b_λ is a sum of products of sums of products of positive elements, and so is also positive.

(iii) [J, (3.25)].

(iv) [J, (3.27), (3.28)].

(v) [J, (3.29)]. ∎

(5.4) THEOREM. *For each conjugacy class C of $W = W(A_n)$ there exist elements $L_C \in Z(\mathscr{H})^+$ satisfying* (L1) *and* (L2) (*see* (4.3)).

Proof. We proceed by reverse induction on the length l_λ of the shortest element of the conjugacy class C_λ.

The longest shortest element of any conjugacy class is that of the Coxeter class, which corresponds to the partition $\lambda = (n+1)$. In this case we have that $b_{(n+1)}$ satisfies (L1) directly from (5.3)(v). By (5.3)(iii) and (iv), the only elements of length n are those from $C_{(n+1),\min}$, and these appear with integer coefficient in $b_{(n+1)}$. Then by (5.3)(v), they in fact all have the same coefficient. All other terms in $b_{(n+1)}$ are strictly longer than n, and not shortest in any conjugacy class, and so $b_{(n+1)}$ also satisfies (L2), and we have the existence of the required L_C.

Suppose inductively that for $l_\lambda > k$ we have the existence of L_{C_λ} with the required properties. Then we also have that Γ_{C_λ} exists for $l_\lambda > k$, and thus we have the results of Section 4 for those conjugacy classes.

Take a conjugacy class C_λ with $l_\lambda = k$. The only terms of length k in the Jones element b_λ are elements of $C_{\lambda,\min}$. If there are shortest elements of other conjugacy classes in b_λ they must have length strictly greater than k. By (4.6), if $r\tilde{T}_w \le b_\lambda$ for $w \in C_{\min}$ for some C, then $r\Gamma_C \le b_\lambda$, and so we may subtract $r\Gamma_C$ from b_λ while remaining in $Z(\mathscr{H})^+$. In this way we may remove all shortest elements other than those of length k from b_λ, giving us a positive central element which specializes to an $\mathbb{N}$-multiple of $\tilde{T}_{C_\lambda}$ and which contains no other shortest elements with non-zero coefficient. In other words, we have an element satisfying (L1) and (L2). This proves the theorem. ∎

(5.5) COROLLARY. *In type A_n we have the following:*

(i) *For any $i \in \mathbb{N}$, $\mathfrak{A}^i(\tilde{T}_C)$ and $\mathfrak{A}^i(\tilde{T}_{C_{\min}})$ are well defined, and there exist finite $n, n' \in \mathbb{N}$ such that $\mathfrak{A}^n(\tilde{T}_C) = \mathfrak{A}^{n'}(\tilde{T}_{C_{\min}}) = \Gamma_C$ satisfies* (G1) *and* (G2) *from* (4.5).

(ii) $Z(\mathscr{H})^+_{\min} = \{\Gamma_C \mid C \in \mathrm{ccl}(W)\}$.

(iii) $Z(\mathscr{H})^+_{\min}$ *is a $\mathbb{Z}[\xi]$-basis for $Z(\mathscr{H})$.*

Proof. (i) follows from (4.5) and (5.3). (ii) follows from (i) and (4.7)(iii). (iii) follows from (5.4) and (4.7). ∎

6. GENERAL WEYL GROUPS

Recent results of Geck and Rouquier [GR] provide the type of element required for the results in Section 4 in order to prove the algorithm is well defined for all types of Weyl groups, and that the primitive minimal positive central elements are a basis for $Z(\mathscr{H})$.

Define the irreducible characters ϕ_i: $\mathscr{H}_{\mathbb{Q}(\xi)} \to \mathbb{Q}(\xi)$. These are $\mathbb{Q}(\xi)$-linear maps with the property that for any $a, b \in \mathscr{H}_{\mathbb{Q}(\xi)}$, $\phi_i(ab) = \phi_i(ba)$. Starkey and Ram (independently in [C2] and [R], respectively) have shown in type A_n that these characters are constant on shortest elements of conjugacy classes. Geck and Pfeiffer extended this to all types of finite Weyl group in [GP] using (1.1).

Since $\mathscr{H}_{\mathbb{Q}(\xi)}$ is isomorphic to $\mathbb{Q}(\xi)W$, we have that the number of irreducible characters is the same as the number of conjugacy classes. Theorem (1.1) shows that we may write the image of any generator $\tilde{T}_w$ under any central function ϕ as an $\mathbb{N}[\xi]$-linear combination of images of shortest elements from conjugacy classes, using the relations $\phi(\tilde{T}_{sds}) = \phi(\tilde{T}_{sd}\tilde{T}_s) = \phi(\tilde{T}_s\tilde{T}_{sd}) = \phi(\tilde{T}_d + \xi\tilde{T}_{sd}) = \phi(\tilde{T}_d) + \xi\phi(\tilde{T}_{sd})$ and $\phi(\tilde{T}_{ds}) = \phi(\tilde{T}_{sd})$ for $d \in \mathfrak{D}_{\langle s\rangle, \langle s\rangle}$. Thus we may write

$$\phi_i(\tilde{T}_w) = \sum_C f_{w,C}\phi_i(\tilde{T}_{w_C})$$

for any $w \in W$, all irreducible characters ϕ_i, and for some $w_C \in C_{\min}$. Further, the $f_{w,C}$ are unique since the irreducible characters are linearly independent (which means the matrix $(\phi_i(\tilde{T}_{w_{C_j}}))$ is invertible).

Geck and Rouquier [GR, Sect. 5] then point out that the functions f_C: $\mathscr{H} \to \mathbb{Z}[\xi]$ defined on the generators by sending $\tilde{T}_w$ to $f_{w,C}$ are central, and that for any central function $\varphi \in CF(\mathscr{H})$ we may write $\varphi = \sum_C \varphi(\tilde{T}_{w_C})f_C$, which means the set $\{f_C\colon C \in \operatorname{ccl}(W)\}$ is a $\mathbb{Z}[\xi]$-basis for $CF(\mathscr{H})$. They then call upon the correspondence between central functions and central elements to get elements z_C which form a $\mathbb{Z}[\xi]$-basis for the centre of $\mathscr{H}$, $z_C = \sum_{w\in W} f_C(\tilde{T}_w)\tilde{T}_{w^{-1}}$. [Note that since Geck and Rouquier work over $\mathbb{Z}[q, q^{-1}]$, they need an additional weighting factor $q^{-l(w)}$ which is not necessary over $\mathbb{Z}[\xi]$.] A key part of their proof is the recognition that for $w_{C'} \in C'_{\min}$, $f_{w_{C'},C} = \delta_{C,C'}$.

Our methods then provide an alternative proof that this set of elements is a basis for the centre:

(6.1) LEMMA. *The set* $\{\sum_{w\in W} f_C(\tilde{T}_w)\tilde{T}_w \mid C \in \operatorname{ccl}(W)\}$ *satisfies properties* (G1) *and* (G2), *and so* $z_C = \Gamma_C$ *for all* C.

Proof. The centrality follows directly from Lemma (3.1) and the relations $\phi(\tilde{T}_{sds}) = \phi(\tilde{T}_d) + \xi\phi(\tilde{T}_{ds})$ and $\phi(\tilde{T}_{ds}) = \phi(\tilde{T}_{sd})$ for any $d \in \mathfrak{D}_{\langle s\rangle, \langle s\rangle}$ and any central function ϕ.

The image of a shortest element of a conjugacy class under f_C is either 1 or 0, so we have that the coefficient of a shortest element $\tilde{T}_{w_{C'}}$ in $\sum_{w\in W} f_C(\tilde{T}_w)\tilde{T}_w$ is 1 if $C' = C$ and 0 otherwise. Thus (G2) is satisfied.

If $w \in C'$ for any conjugacy class C' of W, then $f_C(\tilde{T}_w) = f_C(\tilde{T}_{w_{C'}}) + \xi X$, where X is an $\mathbb{N}[\xi]$-linear combination of images $f_C(\tilde{T}_u)$. So since

$f_C(\tilde{T}_{w_{C'}}) = 0$ unless $C = C'$, and 1 if it does, we have $\sum_{w\in W} f_C(\tilde{T}_w)\tilde{T}_w|_{\xi=0} = \tilde{T}_C$, so (G1) (see (4.5)) is satisfied.

Finally the equality $\sum_{w\in W} f_C(\tilde{T}_w)\tilde{T}_w = \sum_{w\in W} f_C(\tilde{T}_w)\tilde{T}_{w^{-1}}$ follows since the latter also satisfies (G1) and (G2), and so by (4.7)(i) they are the same, and are the element Γ_C. ∎

The existence of these elements then provides us with the means to draw our general conclusion for Weyl groups:

Proof of (2.2). Since we have the existence of Γ_C for all C and all Weyl groups W (from (6.1)), (i) follows from (4.8), and (ii) follows from (4.7). ∎

Remark. In order to use characters to calculate a primitive minimal positive central element, as done in other approaches to the question (for example, [R] and [C2] in type A_n and [GR] for general Weyl groups), one must calculate the coefficient $\phi_C(\tilde{T}_w)$ for every w in W. The algorithm $\mathfrak{A}$ has the advantage that one only calculates coefficients for those terms whose coefficient is non-zero.

7. NON-CRYSTALLOGRAPHIC FINITE COXETER GROUPS

We have shown that the set of primitive minimal positive central elements of an Iwahori–Hecke algebra over a Weyl group has remarkable properties, and may be found by an elementary algorithm. This relies on the Weyl group conjugacy result of Geck and Pfeiffer (Theorem 1.1). To complete the full description for all finite Coxeter groups, it remains to establish the results for the dihedral groups $I(n)$, and the groups of type H_3 and H_4.

It is straightforward to check this in the dihedral groups (see below), and can be done for H_3 by explicit calculation of the conjugacy classes (there are 120 elements in $W(H_3)$, and 10 conjugacy classes, all of size less than or equal to 20)—see [Fr] for details. The group H_4 is more difficult to do this way though, as it has 14,400 elements. Possible approaches to checking H_4 may be through computer programs such as Magma or GAP (which Geck and Pfeiffer used to prove (1.1) for the exceptional Weyl groups), or perhaps by using the embedding into E_8 (see [S]).

Note. Since the submission of this paper Meinolf Geck has told me that in fact they have proved (1.1) in the non-crystallographic case also. Their work can be found in [GHLMP, Sect. 3.2], and uses computer algebra. Thus in the following section, (7.1) is merely "independent" rather than "original."

The Dihedral Groups

We show first that the Geck–Pfeiffer theorem holds for dihedral groups, and then provide the set of primitive minimal positive central elements for the Iwahori–Hecke algebras of the dihedral groups.

The dihedral group of type $I(n)$ has generators $\{s, t\}$ with relations $s^2 = t^2 = (st)^n = 1$. The conjugacy classes split into the case where n is even and the case where n is odd. Let us write $n = 2v$ or $n = 2v + 1$.

(7.1) PROPOSITION. *Theorem* 1.1 *holds for the dihedral groups.*

Proof. Consider first $n = 2v$. The longest word of $I(n)$ is $(st)^v$. There are $v + 3$ conjugacy classes, with representatives $1, s, t, (st)^k$ for $1 \le k \le v$. If we write C_w for the conjugacy class containing $w \in I(n)$, we have

$$\begin{aligned} C_1 &= \{1\}, \\ C_s &= \{(st)^k s, (ts)^l t \mid k \le v - 2 \text{ even}, l \le v - 1 \text{ odd}\}, \\ C_t &= \{(st)^k s, (ts)^l t \mid k \le v - 1 \text{ odd}, l \le v - 1 \text{ even}\}, \\ C_{(st)^k} &= \{(st)^k, (ts)^k\} \qquad \text{for } k < v, \\ C_{(st)^v} &= \{(st)^v\}. \end{aligned}$$

The classes C_1 and $C_{(st)^v}$ are singleton sets, so the theorem trivially holds. The classes $C_{(st)^k}$ for $1 \le k < v$ have only two elements in each, which are both “minimal" in length in the class, so the proposition holds. Finally, every element of C_s and C_t has a shorter conjugate by either s or t, except the minimal element of the class, so the proposition holds here too.

If $n = 2v + 1$, the longest word is $(st)^v s = (ts)^v t$, and there are $v + 2$ conjugacy classes with representatives $1, s, (st)^k$. The classes are

$$\begin{aligned} C_1 &= \{1\}, \\ C_s &= \{(st)^k s, (ts)^l t \mid 0 \le k \le v,\ 0 \le l < v\}, \\ C_{(st)^k} &= \{(st)^k, (ts)^k\} \text{ for } 1 \le k \le v\}. \end{aligned}$$

Again C_1 is trivial, and each $C_{(st)^k}$ contains only two elements of the same length. As with the even case, every element of C_s has an s- or t-conjugate of strictly shorter length, so the proposition holds. ∎

For $\sigma \in S$, we denote the subset of elements in C_σ of length greater than or equal to i by $C_{\sigma, i}$.

Given (7.1), we may use the algorithm $\mathfrak{A}$ to find the primitive minimal positive central elements of the Iwahori–Hecke algebras of the dihedral groups. We provide without proof the following set of elements of $Z(\mathscr{H}(I(n)))^+_{\min}$.

(7.2) THEOREM. *The following set of elements is the set of primitive minimal positive central elements of the Iwahori–Hecke algebra of the dihedral group $I(n)$, and thus forms a $\mathbb{Z}[\xi]$-basis for $Z(\mathscr{H}(I(n)))$:*
$n = 2v$ *even*, $1 \leq k \leq v$,

$$\Gamma_1 = \tilde{T}_{C_1},$$
$$\Gamma_\sigma = \tilde{T}_{C_\sigma} \quad \text{for } \sigma \in S,$$
$$\Gamma_{(st)^k} = \tilde{T}_{C_{(st)^k}} + \xi \sum_{\substack{i>2k \\ \sigma\in S}} \tilde{T}_{C_{\sigma,i}};$$

$n = 2v+1$ *odd*, $1 \leq k \leq v$:

$$\Gamma_1 = \tilde{T}_{C_1},$$
$$\Gamma_s = \tilde{T}_{C_s},$$
$$\Gamma_{(st)^k} = \tilde{T}_{C_{(st)^k}} + \xi \textstyle\sum_{i>2k} \tilde{T}_{C_{s,i}}.$$

These elements may be compared with the similar $\mathbb{Z}[\xi]$-basis for the centre found by Fakiolas (in [Fa]) working over the ring $\mathbb{Q}[q]$, which we denote b_w for w a representative of the conjugacy class C. We have the following relations between the elements in [Fa] (modified to be over $\mathbb{Z}[\xi]$) and those above:

For $n = 2v$ even, $1 \leq k < v$,

$$b_1 = \Gamma_1,$$
$$b_\sigma = \Gamma_\sigma,$$
$$b_{(st)^k} = \Gamma_{(st)^k} - \xi(\Gamma_s + \Gamma_t),$$
$$b_{(st)^v} = \Gamma_{(st)^v};$$

for $n = 2v+1$ odd, $1 \leq k \leq v$,

$$b_1 = \Gamma_1,$$
$$b_s = \Gamma_s,$$
$$b_{(st)^k} = \Gamma_{(st)^k} - \xi\Gamma_s.$$

8. EXAMPLES

8.1. *Type A_2*

The Weyl group of type A_2 is generated by the simple reflections s_1 and s_2 with relations $s_i^2 = (s_1s_2)^3 = 1$. It has six elements, $\{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$, and three conjugacy classes. The conjugacy classes C_i of

$W(A_2)$ and the corresponding primitive minimal positive central elements are

$$\begin{aligned} &C_1 = \{1\}, && C_2 = \{s_1, s_2, s_1s_2s_1\}, && C_3 = \{s_1s_2, s_2s_1\}, \\ &\Gamma_1 = \tilde{T}_{C_1}, && \Gamma_2 = \tilde{T}_{C_2}, && \Gamma_3 = \tilde{T}_{C_3} + \xi\tilde{T}_{s_1s_2s_1}. \end{aligned}$$

8.2. *Explicit Calculation in Type B_2*

The Weyl group of type B_2 has eight elements generated by the simple reflections $\{s, t\}$ with relations $s^2 = t^2 = (st)^4 = 1$. It has five conjugacy classes, but the only primitive minimal positive central element which is not simply the conjugacy class sum corresponds to the class $\{st, ts\}$. We show diagramatically the construction using the algorithm (note that sts commutes with t, and tst with s, so that $\tilde{T}_{sts}$ (resp. $\tilde{T}_{tst}$) is a t-class element (resp. s-class element) on its own):

$$\begin{tikzcd} \tilde{T}_{st} \arrow[d, "t"'] \arrow[dr, "s"] & \tilde{T}_{ts} \arrow[dl, "t"'] \arrow[d, "s"] \\ \xi\tilde{T}_{tst} & \xi\tilde{T}_{sts} \end{tikzcd}$$

8.3. *Type A_3*

Let W be the Weyl group of type A_3, generated by s_1, s_2, and s_3, with relations $s_j^2 = (s_1s_3)^2 = (s_is_{i+1})^3 = 1$ for $j = 1, 2, 3$ and $i = 1, 2$. The conjugacy classes of W are

$$\begin{aligned} C_{\mathrm{id}} &= \{1\}, \\ C_{12} &= \{s_1s_2, s_2s_1, s_2s_3, s_3s_2, s_2s_3s_2s_1, s_1s_2s_3s_2, s_1s_2s_1s_3, s_1s_3s_2s_1\}, \\ C_1 &= \{s_1, s_2, s_3, s_1s_2s_1, s_2s_3s_2, s_1s_2s_3s_2s_1\}, \\ C_{13} &= \{s_1s_3, s_2s_1s_3s_2, s_1s_2s_1s_3s_2s_1\}, \\ C_{123} &= \{s_1s_2s_3, s_2s_1s_3, s_1s_3s_2, s_3s_2s_1, s_1s_2s_1s_3s_2, s_2s_1s_3s_2s_1\}. \end{aligned}$$

For simplicity of notation, we will write $\tilde{T}_{s_i} = \tilde{T}_i$, and as before we will write $\tilde{T}_C$ for the conjugacy class sum. The elements of $Z(\mathscr{H}(A_3))^+_{\min}$ are

$$\begin{aligned} \Gamma_C &= \tilde{T}_C \qquad \text{for } C = C_{\mathrm{id}}, C_1, C_{13}, \\ \Gamma_{C_{12}} &= \tilde{T}_{C_{12}} + \xi(\tilde{T}_{121} + \tilde{T}_{232} + 2\tilde{T}_{12321} + \tilde{T}_{21321} + \tilde{T}_{12132}) + \xi^2\tilde{T}_{121321}, \\ \Gamma_{C_{123}} &= \tilde{T}_{C_{123}} + \xi(\tilde{T}_{1213} + \tilde{T}_{1321} + \tilde{T}_{1232} + \tilde{T}_{2321} + \tilde{T}_{2132} + 2\tilde{T}_{121321}) \\ &\quad + \xi^2(\tilde{T}_{12132} + \tilde{T}_{21321} + \tilde{T}_{12321}) + \xi^3\tilde{T}_{121321}. \end{aligned}$$

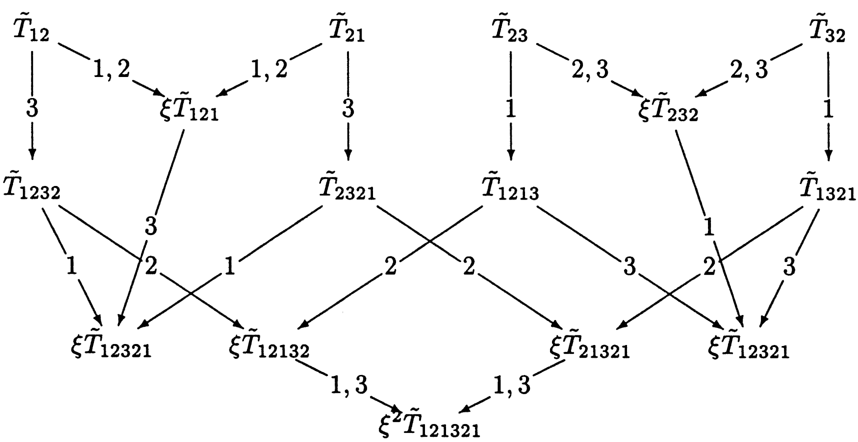
We may compare these elements with the Jones elements (see Section 5), which we denote b_C for the element corresponding to the conjugacy class C:

$$\begin{aligned}
b_{C_{\mathrm{id}}} &= 24\Gamma_{C_{\mathrm{id}}} + 12\xi\Gamma_{C_1} + 6\xi^2\Gamma_{C_{13}} + 4\xi^2\Gamma_{C_{12}} + \xi^3\Gamma_{C_{123}},\\
b_{C_1} &= 2\Gamma_{C_1} + 2\xi\Gamma_{C_{12}} + 2\xi\Gamma_{C_{13}} + \xi^2\Gamma_{C_{123}},\\
b_{C_{13}} &= 2\Gamma_{C_{13}} + \xi\Gamma_{C_{123}},\\
b_{C_{12}} &= \Gamma_{C_{12}} + \xi\Gamma_{C_{123}},\\
b_{C_{123}} &= \Gamma_{C_{123}}.
\end{aligned}$$

The upper-triangularity of these relationships reflects the fact that the Jones element corresponding to a conjugacy class C contains shortest elements only of conjugacy classes of length greater than l_C, apart from those in C (see (5.3)(iii) and (iv)).

8.4. *The Explicit Construction of* $\mathfrak{A}(\tilde{T}_{C_{12}})$ *in Type* A_3

We can graphically show the construction of $\Gamma_{C_{12}}$ in $\mathscr{H}(A_3)$, starting at the top with the shortest elements in C_{12}, and s-class element completions denoted by connecting lines. The practical process is to start with the shortest, and check that for each s there are lines labelled by s connecting the element with the others in its s-class element. The shortest term for which there is no connection for some s is the term we complete.



Note that both $\xi\tilde{T}_{12321}$ and $\xi^2\tilde{T}_{121321}$ are self-conjugate under $\tilde{T}_2$, so form s_2-class elements of Type I.

One can see that for any $s \in S$, we may cut up the above graph into disjoint subgraphs corresponding to the types shown in diagram (3.6) as well as the singleton subgraphs corresponding to Type I s-class elements, although in the above diagram we have suppressed any horizontal lines from the Type II graphs for simplicity. This shows that the sum of the terms above is in the centre, and by checking that each step never adds shorter we have (by (4.5)) that this sum is the element $\Gamma_{C_{12}} \in Z(\mathscr{H}(A_3))^+_{\min}$. Alternatively, to see the sum is $\Gamma_{C_{12}}$ one could observe that it specializes to $\tilde{T}_{C_{12}}$ and that there are no shortest elements from any conjugacy class other than those from $\tilde{T}_{C_{12}}$. We can then make our conclusion using the characterization of (4.7)(i).

8.5. *Type* B_3

Let W be the Weyl group of Type B_3, generated by t, s_1, s_2, and with relations $t^2 = s_i^2 = 1$, $(ts_1)^4 = (s_1s_2)^3 = (ts_2)^2 = 1$. The conjugacy classes of W are

$$C_1 = \{1\},$$

$$C_2 = \{s_2t, s_1s_2ts_1, s_1s_2s_1ts_1s_2, ts_1s_2ts_1t, ts_1s_2s_1ts_1s_2t, s_1ts_1s_2ts_1ts_1\}$$

$$C_3 = \{t, s_1ts_1, s_2s_1ts_1s_2\},$$

$$C_4 = \{s_1, s_2, s_1s_2s_1, ts_1t, ts_1s_2s_1t, s_1ts_1s_2s_1ts_1\},$$

$$C_5 = \{ts_1ts_1, s_2ts_1ts_1s_2, s_1s_2ts_1ts_1s_2s_1\},$$

$$C_6 = \{ts_1ts_1s_2s_1ts_1s_2\},$$

$$C_7 = \{s_1s_2, s_2s_1, ts_2s_1t, ts_1s_2t, s_1s_2s_1ts_1t, ts_1ts_1s_2s_1, ts_2s_1ts_2s_1, s_1s_2ts_1s_2t\},$$

$$C_8 = \{s_1t, ts_1, ts_1s_2s_1, s_1s_2s_1t, s_1ts_1s_2, s_2s_1ts_1\},$$

$$C_9 = \{s_1s_2t, s_2s_1t, ts_1s_2, ts_2s_1, s_2s_1ts_2s_1, s_1s_2ts_1s_2, ts_1ts_2s_1ts_1, s_1ts_1s_2ts_1t\},$$

$$C_{10} = \{s_2s_1ts_1t, s_1s_2ts_1t, ts_1ts_1s_2, ts_1ts_2s_1, s_1s_2s_1ts_2ts_1, s_2ts_1ts_1s_2s_1\}.$$

Again we abbreviate $\tilde{T}_{s_i}$ to $\tilde{T}_i$, and write $\tilde{T}_C$ for the conjugacy class sum. The minimal basis for $Z(\mathscr{H}(B_3))$ over $\mathbb{Z}[\xi]$ is the set $\{\Gamma_1, \ldots, \Gamma_{10}\}$, where the Γ_i are

$$\Gamma_i = \tilde{T}_{C_i} \quad \text{for } i = 1, \ldots, 6,$$

$$\begin{aligned}\Gamma_7 = \tilde{T}_{C_7} + \xi(&\tilde{T}_{121} + \tilde{T}_{t121t} + \tilde{T}_{1t21t12} + \tilde{T}_{t21t121} + 2\tilde{T}_{1t121t1} + \tilde{T}_{t1t121t} \\ &+ \tilde{T}_{t1t21t1}) + \xi^2(\tilde{T}_{1t121t12} + \tilde{T}_{t1t21t12} + \tilde{T}_{t1t121t1}),\end{aligned}$$

$$\begin{aligned}\Gamma_8 = \tilde{T}_{C_8} + \xi(&\tilde{T}_{1t1} + \tilde{T}_{t1t} + 2\tilde{T}_{21t12} + \tilde{T}_{121t1} + \tilde{T}_{1t121} + \tilde{T}_{t121t} + \tilde{T}_{t1t21t1} \\ &+ \tilde{T}_{t1t121t} + \tilde{T}_{1t121t1}) + \xi^2(\tilde{T}_{121t12} + \tilde{T}_{t1t21t12} + \tilde{T}_{t1t121t1}),\end{aligned}$$

$$\begin{aligned}\Gamma_9 = \tilde{T}_{C_9} + \xi(&\tilde{T}_{121t} + \tilde{T}_{t121} + \tilde{T}_{12t1} + \tilde{T}_{21t1} + \tilde{T}_{1t12} + \tilde{T}_{t12t} + \tilde{T}_{t21t} + 2\tilde{T}_{121t12} \\ &+ \tilde{T}_{t1t21t} + \tilde{T}_{1t121t} + \tilde{T}_{t121t1} + \tilde{T}_{t1t121} + \tilde{T}_{121t1t} + 2\tilde{T}_{t1t21t12} + 2\tilde{T}_{t1t121t1})\end{aligned}$$

$$
\begin{aligned}
&+\xi^2(\tilde{T}_{1t121}+\tilde{T}_{121t1}+\tilde{T}_{21t12}+\tilde{T}_{t121t}+2\tilde{T}_{t1t121t}+2\tilde{T}_{t1t21t1}+2\tilde{T}_{1t121t1}\\
&+\tilde{T}_{1t21t12}+\tilde{T}_{t21t121})+\xi^3(\tilde{T}_{121t12}+2\tilde{T}_{t1t21t12}+2\tilde{T}_{t1t121t1}+\tilde{T}_{1t121t12}),\\
\Gamma_{10}=\tilde{T}_{C_{10}}&+\xi(\tilde{T}_{t21t12}+\tilde{T}_{121t1t}+\tilde{T}_{t1t121}+\tilde{T}_{t121t1}+\tilde{T}_{1t121t}+\tilde{T}_{t1t21t}\\
&+\tilde{T}_{t1t21t12}+2\tilde{T}_{1t121t12}+\tilde{T}_{t1t121t1})+\xi^2(\tilde{T}_{1t21t12}+\tilde{T}_{t121t12}\\
&+\tilde{T}_{1t121t1}+\tilde{T}_{t1t21t1}+\tilde{T}_{t1t121t})+\xi^3(\tilde{T}_{1t121t12}+\tilde{T}_{t1t21t12}+\tilde{T}_{t1t121t1}).
\end{aligned}
$$

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