# THE BRAUER HOMOMORPHISM AND THE MINIMAL BASIS FOR CENTERS OF IWAHORI-HECKE ALGEBRAS OF TYPE $\boldsymbol{A}^{*}$ 

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#### Abstract

We use the results on the minimal basis of the centre of an Iwahori-Hecke algebra from our earlier work, as well as some additional results on the minimal basis, to describe the image and kernel of the Brauer homomorphism for Iwahori-Hecke algebras defined by L. Jones (Jones, L. Centres of Generic Hecke Algebras; Ph.D. Thesis; University of Virginia, 1987.).

Key Words: Brauer homomorphism; Iwahori-Hecke algebra; Centre; Minimal basis; Symmetric group


## 0. INTRODUCTION

The Brauer homomorphism is an important tool in the block theory of finite groups. It was first defined by Brauer in the 1950s (1), generalized by Scott in

[^0]1973 (2) to the theory of permutation representations, and has since been defined by Jones in 1987 (3) for Iwahori-Hecke algebras, and by (4) for $q$-Schur algebras.

In the classical theory, Brauer and Robinson (5) used this homomorphism to describe the block theory of the symmetric group algebra, a result known as the Nakayama conjecture. In particular, they needed to know which conjugacy class sums were in its kernel. In the Iwahori-Hecke algebra case, the Nakayama conjecture has been recently proved by James and Mathas (6) in the general setting, using methods along the lines of the proof of Murphy (7) in the classical case, rather than the Brauer-Robinson approach. It is hoped that better understanding of the class elements in the kernel of the Brauer homomorphism may lead to an alternative proof of the Nakayama conjecture for Iwahori-Hecke algebras analogous to the Brauer-Robinson proof in the classical case.

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Let $W$ be a Weyl group of type $A_{n-1}$, with generating set $S=\left\{s_{1}, \ldots, s_{n-1}\right\}$, and length function $l: W \rightarrow \mathbb{N}$. Then for $s_{i}, s_{j} \in S$ and $i \neq j, W$ has relations

$$
\begin{aligned}
s_{i}^{2} & =1 \\
\left(s_{i} s_{j}\right)^{m_{i j}} & =1
\end{aligned}
$$

where $m_{i j}=3$ if $|i-j|=1$ and $m_{i j}=2$ otherwise. Each Weyl group of type $A_{n-1}$ is partitioned into conjugacy classes $C$, which are parameterized by partitions $\lambda$ of $n$. Thus, we will write $C_{\lambda}$ for the conjugacy class corresponding to $\lambda \vdash n$. Let $l_{\lambda}$ be the length of the shortest elements in $C_{\lambda}$, and let $C_{\lambda, \min }$ be the set of shortest elements in $C_{\lambda}$. Let $\operatorname{ccl}(W)$ be the set of conjugacy classes in $W$. For any two parabolic subgroups $W_{1}$ and $W_{2}$ of $W$, write $\mathfrak{D}_{W_{1}, W_{2}}$ for the set of distinguished $W_{1}-W_{2}$ double coset representatives in $W$.

Let $\xi$ be an indeterminant, let $R=\mathbb{Z}[\xi]$, and let $\mathcal{H}$ be the Iwahori-Hecke algebra of $W$ over $R$. Then $\mathcal{H}$ is the associative algebra generated by the set $\left\{\tilde{T}_{s} \mid s \in S\right\}$ with relations

$$
\tilde{T}_{s}^{2}=\tilde{T}_{1}+\xi \tilde{T}_{s}
$$

and

$$
\tilde{T}_{w}=\tilde{T}_{s_{i 1}} \ldots \tilde{T}_{s_{i r}}
$$

when $w=s_{i_{1}} \ldots s_{i_{r}}$ is a reduced expression for $w$. For any subset $X$ of $W$, we will abbreviate the sum of terms corresponding to elements in $X$ by $\tilde{T}_{X}$, that is, $\tilde{T}_{X}:=\sum_{w \in X} \tilde{T}_{w}$. This will most commonly be used in the case $X$ is a conjugacy class.

Define the set $R^{+}:=\mathbb{N}[\xi]$, and let $\mathcal{H}^{+}$be the set of elements of $\mathcal{H}$ whose terms $\tilde{T}_{w}$ all have coefficient in $R^{+}$. That is, $\mathcal{H}^{+}$is the $R^{+}$-span of the set $\left\{\tilde{T}_{w} \mid\right.$ $w \in W\}$.
[Note that $\mathcal{H}$ is a subalgebra of the Iwahori-Hecke algebra $\mathcal{H}_{q}$ over $\mathbb{Z}\left[q^{\frac{1}{2}}\right.$, $\left.q^{-\frac{1}{2}}\right]$ obtained by setting $\xi=q^{\frac{1}{2}}-q^{-\frac{1}{2}}$ and $\tilde{T}_{s}=q^{-\frac{1}{2}} T_{s}$ for each generator $T_{s}$ of $\mathcal{H}_{q}$.]

There is a natural partial ordering on $\mathcal{H}^{+}$, defined by setting $h_{1} \leq h_{2}$ if and only if $h_{2}-h_{1} \in \mathcal{H}^{+}$. We say an element of $\mathcal{H}$ is primitive if its image under the specialisation $\xi=0$ is nonzero.

## 1. THE JONES BRAUER HOMOMORPHISM

In his thesis (3), Lenny Jones defined a Brauer type homomorphism from the centre of the Iwahori-Hecke algebra of type $A_{n-1}$ into the centre of a certain parabolic subalgebra. This Brauer homomorphism has already found applications in the Green correspondence for $\mathcal{H}$-modules in (8).

For $\lambda$ a composition of $n$ (write $\lambda \models n$ ), we write $\mathcal{H}_{\lambda}$ for the subalgebra of $\mathcal{H}$ corresponding to the subgroup $W_{\lambda}$ of $W=W\left(A_{n-1}\right)$. We will be dealing with a few special examples, so we define the following abbreviations. Let $\gamma=(l, n-l) \models n$, and write $W_{l}=W_{\left(l, 1^{n-l}\right)}$ and $\mathcal{H}_{l}=\mathcal{H}_{\left(l, 1^{n-l}\right)}$ for some $l<n$. For clarity, we will continue to write $W_{\left(1^{l}, n-l\right)}$ and $\mathcal{H}_{\left(1^{l}, n-l\right)}$ in full.

The Brauer homomorphism is a composition of two maps. Let $\rho: \mathcal{H} \rightarrow \mathcal{H}_{\gamma}$ be the canonical projection onto the subalgebra $\mathcal{H}_{\gamma}$ of $\mathcal{H}$. As in (9), we may decompose the centralizer of $\mathcal{H}_{\gamma}$ in $\mathcal{H}$ as follows:

$$
Z_{\mathcal{H}}\left(\mathcal{H}_{\gamma}\right)=Z\left(\mathcal{H}_{\gamma}\right) \oplus \bigoplus_{1 \neq x \in \mathfrak{D}_{W_{\gamma}, W_{\gamma}}}\left(Z_{\mathcal{H}_{\gamma} \tilde{T}_{x} \mathcal{H}_{\gamma}}\left(\mathcal{H}_{\gamma}\right)\right)
$$

Thus, since $Z(\mathcal{H})$ is a subalgebra of $Z_{\mathcal{H}}\left(\mathcal{H}_{\gamma}\right), \rho(Z(\mathcal{H}))$ is contained in $Z\left(\mathcal{H}_{\gamma}\right)$. In what follows, we will generally regard $\rho$ as the projection $\mathcal{H} \rightarrow \mathcal{H}_{\gamma}$ restricted to the domain $Z(\mathcal{H})$.

Recall from (9) that the norm map $N_{W, 1}$ for $W$ a Weyl group, takes $h \in \mathcal{H}$ to $\sum_{w \in W} \tilde{T}_{w} h \tilde{T}_{w^{-1}}$.

The Brauer homomorphism $\sigma$ is defined to be the composition of $\rho$ with the canonical homomorphism $\theta: Z\left(\mathcal{H}_{\gamma}\right) \rightarrow Z\left(\mathcal{H}_{\gamma}\right) /\left[N_{W_{l}, 1}\left(\mathcal{H}_{\gamma}\right) \cap Z\left(\mathcal{H}_{\gamma}\right)\right]$, giving us an algebra homomorphism [for proof of this fact, see (3)]:

$$
\sigma: Z(\mathcal{H}) \longrightarrow Z\left(\mathcal{H}_{\gamma}\right) /\left[N_{W_{l}, 1}\left(\mathcal{H}_{\gamma}\right) \cap Z\left(\mathcal{H}_{\gamma}\right)\right]
$$

## 2. THE MINIMAL BASIS

We begin with some results on elements of $W$.

Lemma 2.1. Let $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}$, for $s \in S$, such that $d s \neq s d$. If $s_{i} \in S$, then

$$
\begin{aligned}
s_{i} \leq d & \Longrightarrow s_{i} \leq s d s \\
s_{i} \leq d s & \Longrightarrow s_{i} \leq s d \quad \text { and } \quad s_{i} \leq s d s
\end{aligned}
$$

where the inequality is the Bruhat order.
Proof: We may assume $s_{i}$ is not equal to $s$, as otherwise the lemma follows trivially. Since $s d \neq d s$, we have $l(s d s)=l(d)+2$, so the first implication follows. On the other hand, if $s_{i} \leq d s$, then $s_{i}$ must be less than $d$ in the Bruhat order, and so less than $s d$ and $s d s$, since $l(s d)=l(d)+1$ and $l(s d s)=l(d)+2$.

The contrapositive of the above is actually very useful, so we state it seperately.

Lemma 2.2. Let $d \in \mathfrak{D}_{\langle s\rangle,(s\rangle)}$, for $s \in S$, such that $d s \neq s d$, and let $s_{i} \in S$. Then

$$
\begin{aligned}
s_{i} \not \leq s d s & \Longrightarrow s_{i} \not \leq d, d s, \text { or } s d, \\
s_{i} \not \leq s d & \Longrightarrow s_{i} \nsubseteq d s \text { or } d .
\end{aligned}
$$

The referee of this paper has pointed out that Lemma 2.2 in fact follows directly from Matsumoto's Theorem, see (10), Theorem (1.8).

From (11), we have the existence of a well-defined algorithm that starts from a sum in $H$ of elements of $C_{\text {min }}$ for a conjugacy class $C$, and adds terms which complete " $s$-class elements" finally arriving at an element of the centre called a "class element." The full set of $s$-class elements is the set of all elements of one of the following forms, for $s \in S$ and $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}$ :

$$
\begin{aligned}
\text { Type I, } d \in Z_{W}(s): \quad b_{d}^{I} & =\tilde{T}_{d} \\
b_{d s}^{I} & =\tilde{T}_{d s} \\
\text { Type II, } d \notin Z_{W}(s): \quad b_{d}^{I I} & =\tilde{T}_{d}+\tilde{T}_{s d s} \\
b_{d s}^{I I} & =\tilde{T}_{d s}+\tilde{T}_{s d}+\xi \tilde{T}_{s d s}
\end{aligned}
$$

The set of all $s$-class elements is in fact the set of primitive minimal positive elements of $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$ under the order defined in the Introduction, and is an $R$-basis for the centraliser $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$ (referred to as the "minimal" basis of $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$ ). See (11) for the full details of the minimal basis approach.

Lemma 2.3. Let $h \in Z(\mathcal{H})^{+}$, and suppose $d \in \mathfrak{D}_{\langle s\rangle,\langle s\rangle}$ such that $d s \neq s d$. Then

$$
\begin{aligned}
r \tilde{T}_{d} \leq h & \Longrightarrow r b_{d}^{I I} \leq h \\
r \tilde{T}_{d s} \leq h & \Longrightarrow r b_{d s}^{I I} \leq h \\
r \tilde{T}_{s d} \leq h & \Longrightarrow r b_{d s}^{I I} \leq h
\end{aligned}
$$

Proof: Since $h$ is also an element of $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$, it can be written as a unique linear combination of the $s$-class elements. As the shortest elements of any $s$-conjugacy class appear in exactly one $s$-class element, the lemma follows.
Definition 2.4. A class element $\Gamma_{C}$ is a central element with the properties that $\left.\Gamma_{C}\right|_{\xi=0}=\tilde{T}_{C}$ for $C \in \operatorname{ccl}(W)$, and that $\Gamma_{C}-\tilde{T}_{C}$ contains no shortest elements of any conjugacy class with nonzero coefficient.

The first discussion of elements with the properties in the definition above appearing in the literature (as pointed out by the referee of this paper) is in (12) Lemma (2.13).

The existence of the class elements can be obtained in several ways, including in type $A$ using elements obtained in (9) combined with the algorithm described below [see (11)]. They were first obtained for general Weyl groups in (13) using character theory, and most recently in (14) in a combinatorial way. In addition, it is easy to show that for each $C \in \operatorname{ccl}(W)$, the class element $\Gamma_{C}$ is unique [see (11), 4.7].

We have from (11) the following:
Theorem 2.5. The set of primitive minimal positive central elements in $\mathcal{H}$, $Z(\mathcal{H})_{\min }^{+}$is equal to $\left\{\Gamma_{C} \mid C \in \operatorname{ccl}(W)\right\}$ and is an $R$-basis for the centre $Z(\mathcal{H})$.

This theorem (as well as the theorem resulting from the Geck-Rouquier approach) also holds for centres of parabolic subalgebras of $\mathcal{H}$, and we will occasionally need to use this fact.

We will briefly describe the functioning of the algorithm mentioned previously.

Suppose $h=h_{s}+h_{s}^{\prime} \in \mathcal{H}^{+}$, with $h_{s}$ a maximal element of $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$ less than $h$. Let $m_{s}$ be the length of the shortest term in $h_{s}^{\prime}$ (for nonzero $h_{s}^{\prime}$ ).
Definition 2.6. Let $h \in \mathcal{H}^{+}$. Define the algorithm $\mathfrak{A}$ to conduct the following procedures.
(i) split $h$ into $h=h_{s}+h_{s}^{\prime}$ for each $s \in S$ such that $h_{s}$ is maximal in $Z_{\mathcal{H}}\left(\tilde{T}_{s}\right)$ less than or equal to $h$;
(ii) if $h_{s}^{\prime}=0$ for all $s \in S$, stop;
(ii) ${ }^{\prime}$ otherwise evaluate $m_{s}$ for each $s$ such that $h_{s}^{\prime} \neq 0$, and choose $s \in S$ such that $m_{s}$ is minimal;
(iii) add terms to $h$ which complete the $s$-class elements of those terms in $h_{s}^{\prime}$ of length $m_{s}$;
(iv) declare the new element to be $\mathfrak{A}(h)$, and repeat from (i) with the new element.
The main result of (11) relating to this algorithm is that if we start with the sum of shortest elements in a conjugacy class $\tilde{T}_{C_{\text {min }}}$, then at each step the elements of length $m_{s}$ in $h_{s}^{\prime}$ are all shortest in their $s$-conjugacy class, so that the
$s$-class element containing them is uniquely determined, and all additions are of length $m_{s}$ or longer. This then implies that there is a finite integer $n \in \mathbb{N}$ such that $\mathfrak{A}^{n}\left(\tilde{T}_{C_{\text {min }}}\right)=\Gamma_{C}$.

In practical terms the best way to use the algorithm is to start from the shortest elements in $h$, check they are all in complete $s$-class elements for all $s \in S$, and then move to the next length up. When one comes to a term not in a complete $s$ class element for some $s$, we add the required terms. This is especially convenient when one has started the algorithm on a conjugacy class sum, as one then knows that the additions are always longer or same-length, and that therefore one does not need to recheck all the short terms at each step. That is, after filling the $s$-class elements of a term of length $l$, one simply looks for any other terms of length $l$ in incomplete $s$-class elements and completes those. If there are none, one moves to terms of length $l+1$.

The algorithm is more than just a computational tool, as it allows us to derive certain properties of the class elements. The rest of this section is devoted to extracting some necessary properties for the later Brauer homomorphism results.

Lemma 2.7. If $w \in C_{\min }$ and $r \tilde{T}_{w} \leq h \in Z(\mathcal{H})^{+}$for some $r \in R^{+}$, then $r \Gamma_{C} \leq h$.

Proof: We may write $h$ as a unique $R$-linear combination of the class elements, by (2.5). However since $h$ is positive, the coefficients of shortest elements in conjugacy classes are positive, and these are found in unique class elements with coefficient one. Thus, the coefficient of a class element in $h$ is the same as that of its corresponding shortest elements, and thus positive. The lemma follows.

If a term $r \tilde{T}_{w}$ appears with $r \neq 0$ in a class element $\Gamma_{C}$, it is probably clear what is meant when we say " $r \tilde{T}_{w}$ arises from $\tilde{T}_{C_{\min }}$ " in the context of additions via the algorithm $\mathfrak{A}$. We want to define this more explicitly.
Definition 2.8. Let $w \in W$, and $u \in C_{\min }$. Then $\tilde{T}_{w}$ is said to arise from $\tilde{T}_{u}$ if there exists a sequence $v_{0}, v_{1}, \ldots, v_{m}$ of elements of $W$ with $u=v_{0}$ and $w=v_{m}$, and a sequence of simple reflections $s_{j_{0}}, \ldots, s_{j_{m-1}}$ from $S$ such that for each $i$ and $d_{i}$ distinguished in $\left\langle s_{j_{i}}\right\rangle d_{i}\left\langle s_{j_{i}}\right\rangle$, we have $l\left(s_{j_{i}} d_{i} s_{j_{i}}\right)=l\left(d_{i}\right)+2$ and either:

1. $v_{i}=d_{i}$ and $v_{i+1}=s_{j_{i}} d_{i} s_{j_{i}}$, or
2. $v_{i}=d_{i} s_{j_{i}}$ and $v_{i+1}=s_{j_{i}} d_{i}$ or $s_{j_{i}} d_{i} s_{j_{i}}$, or
3. $v_{i}=s_{j_{i}} d_{i}$ and $v_{i+1}=d_{i} s_{j_{i}}$ or $s_{j_{i}} d_{i} s_{j_{i}}$.

Lemma 2.9. Suppose $\tilde{T}_{w}$ arises from $\tilde{T}_{u}$ with $u \in C_{\min }$, and let $v_{0}, \ldots, v_{m}$ be the sequence linking $u$ and $w$, with $s_{j_{0}}, \ldots, s_{j_{m-1}}$ the corresponding sequence of simple reflections. Then
(i) $s_{j_{k}} \leq v_{k+1}$ for all $0 \leq k<i \leq m-1$, and
(ii) if $s_{l} \in S \backslash\left\{s_{j_{i}} \mid 0 \leq i \leq m-1\right\}$, then $s_{l} \not \leq w$ implies that $s_{l} \not \leq u$.

Proof: (i) By induction on either $m$ or $k$, and using Lemma 2.1. (ii) follows from the contrapositive of (i).
Proposition 2.10. Let $C$ be a conjugacy class in $W, \Gamma_{C}$ be the corresponding class element, and $s \in S$. Then $s \leq u$ for all $u \in C_{\min }$ implies $s \leq w$ whenever $r \tilde{T}_{w} \leq \Gamma_{C}$ for some $r \in R^{+}$. Similarly, $s \not \leq w$ for some $r \tilde{T}_{w} \leq \Gamma_{C}$ implies $s \not \leq u$ for some $\tilde{T}_{u} \leq \tilde{T}_{C_{\text {min }}}$.
Proof: It is clear from the definition that $\tilde{T}_{w}$ appears in $\Gamma_{C}$ with nonzero coefficient if and only if it arises from $\tilde{T}_{u}$ for some $u \in C_{\min }$. The key observation here is that in the sequence of $v_{i}$ described above, if a simple reflection $s$ is less than $v_{i}$ then it is also less than $v_{i+1}$, by (2.1). And so, if $s \leq u \in C_{\min }$, then $s$ is less than every term that arises from $\tilde{T}_{u}$. On the other hand, if $s$ is not less than $w$ in some term $r \tilde{T}_{w}$ in $\Gamma_{C}$, then there must be a term $\tilde{T}_{u}$ in $\tilde{T}_{C_{\text {min }}}$ from which $r \tilde{T}_{w}$ arose and that also does not contain $s$.

## 3. THE IMAGE OF THE MINIMAL BASIS

We begin by noting a straightforward lemma.
Lemma 3.1. Let $C$ be a conjugacy class of $W_{\gamma}$ where $\gamma=(l, n-l)$. Then $C=$ $C_{1} C_{2}$ where $C_{1}$ is a conjugacy class in $W_{l}$ and $C_{2}$ is a conjugacy class in $W_{\left(1^{l}, n-l\right)}$. Similarly, if $\Gamma_{C}$ is a class element in $\mathcal{H}_{\gamma}$, then $\Gamma_{C}=\Gamma_{C_{1}} \Gamma_{C_{2}}$.
Proof: This follows by noting that $W_{l}$ and $W_{\left(1^{l}, n-l\right)}$ commute, so conjugation in $W_{\gamma}$ is actually two separate commuting conjugations, of the two separate components of any element of $W_{\gamma}$.

For the $\xi$-analogue, clearly $\Gamma_{C_{1}} \Gamma_{C_{2}} \in Z\left(\mathcal{H}_{\gamma}\right)$. It is also easy to see that $\Gamma_{C_{1}} \Gamma_{C_{2}}$ specializes to $\tilde{T}_{C}$ and contains no other shortest elements of any $W_{\gamma^{-}}$ conjugacy class. Then by the uniqueness of the class elements, $\Gamma_{C_{1}} \Gamma_{C_{2}}=\Gamma_{C}$.

There are some useful characteristics of the minimal basis that help us describe its image under $\sigma$. One of these in particular is that expressed in Proposition (2.10), that if a generator ( $s_{l}$ for example) does not appear in a term in a class element, then the term has arisen via additions in the algorithm from a minimal element also without the generator $s_{l}$. This gives us:

Lemma 3.2. Let $C$ be a conjugacy class of $W$. The image of $\Gamma_{C}$ under the projection $\rho$ is a sum of class elements of $\mathcal{H}_{\gamma}$, each with coefficient one.

Proof: The image $\rho\left(\Gamma_{C}\right)$ must be an element of $Z\left(\mathcal{H}_{\gamma}\right)$, and so a linear combination of class elements of $\mathcal{H}_{\gamma}$. Now suppose one of these class elements of $\mathcal{H}_{\gamma}$ has coefficient not equal to one, and let $C_{\gamma}$ be the corresponding conjugacy class in $W_{\gamma}$. It suffices to show that $C_{\gamma, \min } \subseteq C_{\min }$.

Let $u \in C_{\gamma, \min }$, and suppose $u$ is not in $C_{\min }$. Then $u$ has arisen in $\Gamma_{C}$ from a shortest element of $C$ via a sequence of reflections $s_{0}, \ldots, s_{m}$. If this sequence did not include $s_{l}$, then the element giving rise to $u$ from $C_{\min }$ would have to be in $W_{\gamma}$ also, and this would mean that $u$ could not be minimal in $C_{\gamma}$. Thus, the sequence $s_{0}, \ldots, s_{m}$ must include $s_{l}$. Then by Definition (2.8) we must have $s_{l} \leq u$, a contradiction.

If $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots\right) \vdash n$, then we call the elements $\lambda_{i}$ the components of $\lambda$. An $(l, n-l)$-bipartition is a pair of partitions $(\mu, v)$, with $\mu \vdash l$ and $v \vdash n-l$. An $(l, n-l)$-bipartition of $\lambda$ is an $(l, n-l)$-bipartition whose components are those of $\lambda$. Let $\operatorname{Bip}(\lambda)=\left\{\left(\mu_{i}, \nu_{i}\right)\right\}$ be the set of $(l, n-l)$-bipartitions of $\lambda \models n$.

An element $w \in C_{\lambda}$ is said to be of shape $\lambda=\left(\lambda_{1}, \ldots, \lambda_{r}\right)$ if it is in $C_{\lambda, \min }$ and also in $W_{\left(\lambda_{1}, 1^{n-\lambda_{1}}\right)} \times W_{\left(1^{\lambda_{1}}, \lambda_{2}, 1^{\left.n-\lambda_{1}-\lambda_{2}\right)}\right.} \times \cdots \times W_{\left(1^{n-\lambda_{r}}, \lambda_{r}\right)}$.

Proposition 3.3. The number of conjugacy classes of $W_{\gamma}$ contained in $C_{\lambda} \in$ $\operatorname{ccl}(W)($ for $\lambda \models n)$ is the number of $(l, n-l)$-bipartitions of $\lambda$. This is also thus the number of class elements of $\mathcal{H}_{\gamma}$ contained in the class element $\Gamma_{C_{\lambda}}$ of $\mathcal{H}$.
Proof: Firstly note that since $W_{\gamma}=W_{l} \times W_{n-l}$, the conjugacy classes of $W_{\gamma}$ are parametrized by $(l, n-l)$-bipartitions. If $\mu \vdash l$ and $v \vdash n-l$ then we will write $(\mu, \nu)$ for the $(l, n-l)$-bipartition, and $C_{(\mu, \nu)}$ for the corresponding $W_{\gamma}$-conjugacy class.

If $C_{(\mu, \nu)} \subseteq C_{\lambda}$, then any $w \in C_{(\mu, \nu)}$ is of cycle type $\lambda$, and so ( $\mu, \nu$ ) must be a bipartition of $\lambda$. On the other hand if $(\mu, \nu)$ is a bipartition of $\lambda$, then every element of $C_{(\mu, \nu)}$ must also be in $C_{\lambda}$.

Corollary 3.4. Let $\lambda \models n$. Then

$$
\rho\left(\Gamma_{C_{\lambda}}\right)=\sum_{\left(\mu_{i}, v_{i}\right) \in \operatorname{Bip}(\lambda)} \Gamma_{C_{\mu_{i}}} \Gamma_{C_{v_{i}}}
$$

where $C_{\mu_{i}}$ and $C_{v_{i}}$ are conjugacy classes in $W_{l}$ and $W_{\left(1^{l}, n-l\right)}$, respectively.
Proof: This follows from Lemma (3.2) and Proposition (3.3).
Lemma 3.5. Let $w \in C_{\min }$ for some conjugacy class $C$ of $W_{l}$ for $l>1$. Then $\Gamma_{C}<N_{W_{l}, 1}\left(\tilde{T}_{w}\right)$.

Proof: Firstly, we know $\Gamma_{C} \leq N_{W_{l}, 1}\left(\tilde{T}_{w}\right)$, since $N_{W_{l}, 1}\left(\tilde{T}_{w}\right)$ is positive and both central in $\mathcal{H}_{l}$ and contains shortest elements of $C$, so satisfies the requirements of Lemma (2.7). So it suffices to show that either the shortest elements of other
conjugacy classes also occur in $N_{W_{l}, 1}\left(\tilde{T}_{w}\right)$, or the coefficient of $\tilde{T}_{C}$ in $N_{W_{l}, 1}\left(\tilde{T}_{w}\right)$ is more than one.

Suppose that $w$ is a Coxeter element of $W_{l}$; then $C$ is the Coxeter class. Then the element $N_{W_{l-1}, 1}\left(\tilde{T}_{w}\right)$ satisfies the requirements of Definition (2.4) by [(9), 3.23], so is in fact the class element $\Gamma_{C}$ Then

$$
\begin{aligned}
N_{W_{l}, 1}\left(\tilde{T}_{w}\right) & =N_{W_{l}, W_{l-1}}\left(N_{W_{l-1}, 1}\left(\tilde{T}_{w}\right)\right) \\
& =N_{W_{l}, W_{l-1}}\left(\Gamma_{C}\right) \\
& >\Gamma_{C}
\end{aligned}
$$

where the first equality is by [(9), 2.12].
If $w$ is not a Coxeter element of $W_{l}$ (and so $C$ is not the Coxeter class), then $w$ is a Coxeter element of some parabolic subgroup $W_{\alpha}$ of $W_{l}$ (where $\alpha$ is a partition of $l)$. We write $\alpha=\left(\alpha_{1}, \ldots, \alpha_{t}\right)$. The Jones basis element of the centre of $\mathcal{H}_{l}$ corresponding to this conjugacy class of $W_{l}$ is

$$
b_{\alpha}=N_{W_{l}, W_{\alpha}}\left(\eta_{\alpha}\right)
$$

where

$$
\eta_{\alpha}=\prod_{i=1}^{t} N_{W_{\left(1, \ldots, 1, \alpha_{i}-1,1, \ldots, 1\right)}, 1}\left(\tilde{T}_{w_{i}}\right),
$$

and where $w_{i}$ is the Coxeter element of the component $W_{\alpha_{i}}$ of the parabolic subgroup $W_{\alpha}=W_{\alpha_{1}} \times \cdots \times W_{\alpha_{t}}$.

But then $\eta_{\alpha} \leq N_{W_{\alpha}, 1}\left(\tilde{T}_{w}\right)$, so

$$
\begin{aligned}
N_{W_{l}, 1}\left(\tilde{T}_{w}\right) & =N_{W_{l}, W_{\alpha}}\left(N_{W_{\alpha}, 1}\left(\tilde{T}_{w}\right)\right) \\
& \geq N_{W_{l}, W_{\alpha}}\left(\eta_{\alpha}\right) \\
& =b_{\alpha} .
\end{aligned}
$$

By $[(9), 3.29],\left.b_{\alpha}\right|_{\xi=0}=\left[N_{W_{l}}\left(W_{\alpha}\right): W_{\alpha}\right] \tilde{T}_{C}$, that is, the integer part of the coefficient of $\tilde{T}_{C}$ in $b_{\alpha}$ is [ $N_{W_{l}}\left(W_{\alpha}\right): W_{\alpha}$ ] (Jones further shows that this is the only coefficient of $\tilde{T}_{C}$ in $b_{\alpha}$ in [(9), (3.28), (3.30)], but we do not need this here). Since we have assumed $C=C_{\alpha}$ is not the Coxeter class, we have that $\left[N_{W_{l}}\left(W_{\alpha}\right): W_{\alpha}\right]>1$, and so the coefficient of $\tilde{T}_{C}$ in $N_{W_{l}, 1}\left(\tilde{T}_{w}\right)$ is more than one, and so we must have $N_{W_{l}, 1}\left(\tilde{T}_{w}\right)>\Gamma_{C}$.

## 4. THE KERNEL OF THE BRAUER HOMOMORPHISM

Lemma 4.1. For any $h \in \mathcal{H}_{l}$, and $C \in \operatorname{ccl}\left(W_{\left(1^{l}, n-l\right)}\right)$, the product $N_{W_{l}, 1}(h) \Gamma_{C}$ is in $\operatorname{ker} \theta$.

Proof: Since the conjugations by elements of $\mathcal{H}_{l}$ in the definition of $N_{W_{l}, 1}$ all commute with $\Gamma_{C} \in \mathcal{H}_{\left(1^{l}, n-l\right)}$, we have $N_{W_{l}, 1}(h) \Gamma_{C}=N_{W_{l}, 1}\left(h \Gamma_{C}\right) \in Z\left(\mathcal{H}_{\gamma}\right) \cap$ $N_{W_{l}, 1}\left(\mathcal{H}_{\gamma}\right)$.

Now we wish to describe a little more closely the set $N_{W_{l}, 1}\left(\mathcal{H}_{l}\right)$. This is clearly spanned by the set of norms $\left\{N_{W_{l}, 1}\left(\tilde{T}_{w}\right) \mid w \in W_{l}\right\}$ by the linearity of the norm. However, we can further restrict this spanning set to the set of norms of shortest elements of conjugacy classes $\left\{N_{W_{l}, 1}\left(\tilde{T}_{w_{\mu}}\right) \mid w_{\mu} \in C_{\mu}, \mu \vdash l\right\}$, where $w_{\mu}$ is some fixed shortest element of $C_{\mu}$. This set is also linearly independent (as each specializes to a multiple of a conjugacy class sum, see $[(14),(5.2)])$, and so is an $R$-basis for $N_{W_{l}, 1}\left(\mathcal{H}_{l}\right)$.

Let us define elements $a_{\mu, \mu_{j}} \in R$ by writing

$$
N_{W_{l}, 1}\left(\tilde{T}_{w_{\mu}}\right)=\sum_{\mu_{j} \vdash l} a_{\mu, \mu_{j}} \Gamma_{\mu_{j}},
$$

where we abbreviate $\Gamma_{C_{\mu_{j}}}$ to $\Gamma_{\mu_{j}}$.
If $\mu \vdash l$ and $v \vdash n-l$, then we write $\Gamma_{(\mu)(\nu)}$ for the class element of $Z\left(\mathcal{H}_{\gamma}\right)$ corresponding to the conjugacy class in $W_{\gamma}$ which corresponds to the pair $\mu$ and $\nu$. In other words, $\Gamma_{(\mu)(\nu)}$ is the product of the class elements $\Gamma_{\mu}$ in $\mathcal{H}_{l}$ and $\Gamma_{\nu}$ in $\mathcal{H}_{\left(1^{1}, n-l\right)}$.

We then have the following description of the kernel of $\theta$.
Lemma 4.2. The set $\left\{\sum_{\mu_{j} \vdash l} a_{\mu, \mu_{j}} \Gamma_{\left(\mu_{j}\right)\left(v_{k}\right)} \mid \mu \vdash l, v_{k} \vdash n-l\right\}$ spans $\operatorname{ker} \theta$.
Proof: Certainly all elements of the above form are in the kernel:

$$
\begin{aligned}
\theta\left(\sum_{\mu_{j} \vdash l} a_{\mu, \mu_{j}} \Gamma_{\left(\mu_{j}\right)\left(v_{k}\right)}\right) & =\theta\left(\left(\sum_{\mu_{j} \vdash l} a_{\mu, \mu_{j}} \Gamma_{\mu_{j}}\right) \Gamma_{v_{k}}\right) \\
& =\theta\left(N_{W_{l}, 1}\left(\tilde{T}_{w_{\mu}}\right) \Gamma_{\nu_{k}}\right) \\
& =\theta\left(N_{W_{l}, 1}\left(\tilde{T}_{w_{\mu}} \Gamma_{v_{k}}\right)\right) \\
& \in \theta\left(N_{W_{l}, 1}\left(\mathcal{H}_{\gamma}\right) \cap Z\left(\mathcal{H}_{\gamma}\right)\right) .
\end{aligned}
$$

On the other hand, if $h \in \operatorname{ker} \theta$, then $h \in N_{W_{l}, 1}\left(\mathcal{H}_{\gamma}\right) \cap Z\left(\mathcal{H}_{\gamma}\right)$, and so is of the form $N_{W_{l}, 1}\left(h^{\prime}\right)$ for some $h^{\prime} \in \mathcal{H}_{\gamma}$. But any $h^{\prime} \in \mathcal{H}_{\gamma}$ may be written as a product $h^{\prime}=$ $h_{1}^{\prime} h_{2}^{\prime}$ for $h_{1}^{\prime} \in \mathcal{H}_{l}$ and $h_{2}^{\prime} \in \mathcal{H}_{\left(1^{l}, n-l\right)}$, and since $\mathcal{H}_{l}$ and $\mathcal{H}_{\left(1^{l}, n-l\right)}$ commute, we have

$$
\begin{aligned}
h & =N_{W_{l}, 1}\left(h^{\prime}\right) \\
& =N_{W_{l}, 1}\left(h_{1}^{\prime}\right) h_{2}^{\prime} .
\end{aligned}
$$

Now $N_{W_{l}, 1}\left(h_{1}^{\prime}\right)$ is a linear combination of the $N_{W_{l}, 1}\left(\tilde{T}_{w_{\mu}}\right)$ where $\mu \vdash l$ and $w_{\mu} \in C_{\mu, \min }$, and since $h_{2}^{\prime}$ is in the centre of $\mathcal{H}_{\left(1^{l}, n-l\right)}$ it is a linear combination of the class elements $\Gamma_{v_{k}}$ of $\mathcal{H}_{\left(1^{l}, n-l\right)}$. So $h$ is a linear combination of elements of form $N_{W_{l}, 1}\left(\tilde{T}_{w_{\mu}}\right) \Gamma_{\nu_{k}}=\left(\sum_{\mu_{j} \vdash l} a_{\mu, \mu_{j}} \Gamma_{\mu_{j}}\right) \Gamma_{\nu_{k}}=\sum_{\mu_{j} \vdash l} a_{\mu, \mu_{j}} \Gamma_{\left(\mu_{j}\right)\left(v_{k}\right)}$ for $\mu \vdash l$ and $v_{k} \vdash n-l$.

We wish to describe those elements of $Z(\mathcal{H})$ which map under $\rho$ to the kernel of $\theta$. The kernel of $\sigma$ will then consist of those elements together with those in the kernel of $\rho$. We will consider which elements of $\operatorname{ker} \theta$ are contained in $\rho(Z(\mathcal{H}))$. By the above lemma, any element of $\operatorname{ker} \theta$ may be written

$$
\sum_{\substack{\mu_{i} \vdash l \\ \nu_{k} \vdash-n-l}} r_{i k}\left(\sum_{\mu_{j} \vdash l} a_{\mu_{i}, \mu_{j}} \Gamma_{\left(\mu_{j}\right)\left(v_{k}\right)}\right)=\sum_{\substack{\mu_{i}, \mu_{j} \vdash l \\ v_{k} \vdash n-l}} r_{i k} a_{\mu_{i}, \mu_{j}} \Gamma_{\left(\mu_{j}\right)\left(v_{k}\right)}
$$

for $r_{i k} \in R$.
Lemma 4.3. Let $h=\sum_{\mu_{i}, \mu_{j} \vdash l, v_{k} \vdash-n-l} r_{i k} a_{\mu_{i}, \mu_{j}} \Gamma_{\left(\mu_{j}\right)\left(v_{k}\right)} \in \operatorname{ker} \theta$. Then $h \in \rho(Z(\mathcal{H}))$ if and only if

$$
\sum_{\mu_{i} \vdash l} r_{i k_{1}} a_{\mu_{i}, \mu_{j_{1}}}=\sum_{\mu_{i} \vdash l} r_{i k_{2}} a_{\mu_{i}, \mu_{j_{2}}}
$$

for all pairs of pairs $\left(\mu_{j_{1}}, v_{k_{1}}\right)$ and $\left(\mu_{j_{2}}, v_{k_{2}}\right)$ in $\operatorname{Bip}(\lambda)$ and $\lambda \vDash n$.
Proof: Observe that

$$
\begin{aligned}
h & =\sum_{\substack{\mu_{i}, \mu_{j} \vdash l \\
v_{k} \vdash n-l}} r_{i k} a_{\mu_{i}, \mu_{j}} \Gamma_{\left(\mu_{j}\right)\left(v_{k}\right)} \\
& =\sum_{\substack{\mu_{j} \vdash l \\
v_{k} \vdash n-l}}\left(\sum_{\mu_{i} \vdash l} r_{i k} a_{\mu_{i}, \mu_{j}}\right) \Gamma_{\left(\mu_{j}\right)\left(v_{k}\right)} .
\end{aligned}
$$

Write $(\mu, \nu)$ for the partition of $n$ obtained from the components of $\mu \vdash l$ and $v \vdash n-l$. Then $h$ is in the image of $\rho$ if and only if the coefficients of $\Gamma_{\left(\mu_{j}\right)\left(v_{k}\right)}$ for each $\mu_{j}$ and $\nu_{k}$ corresponding to the same partition $\left(\mu_{j}, v_{k}\right)$ of $n$ are the same [see Corollary (3.4)]. The lemma follows.

We already know that for any $\lambda \models n$ which has no $\gamma$-bipartitions, the class element $\Gamma_{\lambda}$ is in the kernel of $\rho$ and therefore $\sigma$. We can now deal with linear combinations of those class elements corresponding to partitions of $n$ that do have $\gamma$-bipartitions.

Corollary 4.4. The sum $\sum_{\lambda \models n, B i p(\lambda) \neq \emptyset} a_{\lambda} \Gamma_{\lambda}$ with $a_{\lambda} \in R$ is in ker $\sigma$ if and only if for each $\lambda \models n$ there exists a set of elements $\left\{r_{i k} \in R\right\}$ with the property that

$$
a_{\lambda}=\sum_{\mu_{i} \vdash l} r_{i k} a_{\mu_{i}, \mu_{j}}=\sum_{\mu_{i} \vdash l} r_{i k^{\prime}} a_{\mu_{i}, \mu_{j^{\prime}}}
$$

for any two $\gamma$-bipartitions $\left(\mu_{j}, v_{k}\right)$ and $\left(\mu_{j^{\prime}}, \nu_{k^{\prime}}\right)$ of $\lambda$.
Finally we are able to make the following conclusion about the kernel of the Brauer homomorphism.

Theorem 4.5. Let $A$ be the set of all linear combinations of class elements as defined in the above corollary, and let $B$ be the set of all class elements whose corresponding partition of $n$ has no $\gamma$-bipartitions. Then

$$
\operatorname{ker} \sigma=\operatorname{span}\{A \cup B\}
$$

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