

## Centralizers of Iwahori–Hecke Algebras II: The General Case

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Received 23 May 2001

Communicated by Jie Xiao

**Abstract.** This paper is a sequel to [4]. We establish the minimal basis theory for the centralizers of parabolic subalgebras of Iwahori–Hecke algebras associated to finite Coxeter groups of any type, generalizing the approach introduced in [3] from centres to the centralizer case. As a pre-requisite, we prove a reducibility property in the twisted  $J$ -conjugacy classes in finite Coxeter groups, which is a generalization of results in [7] and [4].

**2000 Mathematics Subject Classification:** 20C08, 20F55

**Keywords:** Hecke algebra, Coxeter group, conjugacy class, centralizer, minimal basis

### 1 Introduction

The minimal basis theory of [3] for centres of Iwahori–Hecke algebras was part of a minor flurry of work on the question, including work of Jones [10] and Geck and Rouquier [8], and provided a combinatorial approach to some kind of natural/canonical basis for the centres of these algebras. The current paper completes the generalization of this approach to centralizers of parabolic subalgebras, which was begun in [4] with a minimal basis theory for any centralizer of a parabolic subalgebra in types  $A_n$  and  $B_n$ .

What this generalization to arbitrary finite Coxeter groups requires, by Theorem 1.1 below, is the verification of a pair of properties to be satisfied by the  $J$ -conjugacy classes in the Coxeter group. These properties are:

**Property I:** All  $J$ -conjugacy classes are reducible.

**Property II:** For any  $J$ -conjugacy class  $\mathfrak{C}$  and  $w, w' \in \mathfrak{C}_{\min}$ , we have  $w \sim_{GP} w'$ .

The theorem depending on these is as follows.

**Theorem 1.1.** [4, (4.4), (4.5)] *If  $(W, S)$  is a finite Coxeter system, and  $J \subseteq S$  satisfies Properties I and II, then the set of  $J$ -class elements is the set of primitive minimal positive elements of  $Z_{\mathcal{H}}(\mathcal{H}_J)$ , i.e.,*

$$Z_{\mathcal{H}}(\mathcal{H}_J)_{\min}^+ = \{\Gamma_{\mathfrak{C}} \mid \mathfrak{C} \in \text{ccl}_J(W)\}.$$

Furthermore,  $Z_{\mathcal{H}}(\mathcal{H}_J)_{\min}^+$  is an  $R$ -basis for  $Z_{\mathcal{H}}(\mathcal{H}_J)$ , where  $R = \mathbb{Z}[\xi_s]_{s \in S}$  is the base ring of the Iwahori–Hecke algebra  $\mathcal{H}$ .

Properties I and II are somewhat technical and combinatorial results in the Coxeter group, and proofs of particular cases (the fundamental one being [7]) have so far required case by case analyses and the use of the computer algebra packages GAP [11] and CHEVIE [5]. For the results in this paper, we do not directly use any computer calculations, but nevertheless we rely on other theorems (in [7] and [6]) which do rely on case by case checks.

In [4], it is also shown that Properties I and II depend on the reducibility of certain *twisted*  $J$ -conjugacy classes. While our interest in these is motivated by an application to bases of centralizers in Iwahori–Hecke algebras, recently at least two other papers [1, 6] have appeared studying twisted conjugacy classes for other reasons in different ways.

In [6], Geck, Kim, and Pfeiffer study minimal elements of twisted conjugacy classes in finite Coxeter groups in an effort to extend the results of [7] by finding character tables of Iwahori–Hecke algebras with an associated automorphism fixing the set  $S$  of simple reflections (a “twist”). This work is motivated by Digne and Michel [2] on  $\mathcal{L}$ -functions, and contains an analog of a reducibility result from [7] for twisted conjugacy classes which we will use in the current paper.

Weyl groups with automorphisms which fix the set of generators have also arisen in the study of the group of symmetries of an Artin system in Crisp [1].

## 2 Background Definitions

Let  $W$  be a finite Coxeter group with generating set  $S$  of simple reflections and length function  $l : W \rightarrow \mathbb{N}$  defined by  $S$ . Then for  $s, s' \in S$ , we have  $s^2 = 1$  and  $(ss')^{m_{ss'}} = 1$  for some  $m_{ss'} \in \mathbb{N}$ . For general information about the structure of finite Coxeter groups, the reader is referred to [9].

For  $J \subseteq S$ , each finite Coxeter group is partitioned into  $J$ -conjugacy classes, corresponding to sets of elements conjugate by elements of  $W_J$  (the parabolic subgroup generated by  $J$ ).

Write  $\mathcal{D}_{K,J}$  for the set of distinguished representatives of the double cosets  $W_K d W_J$ , where  $J, K \subseteq S$ .

Let  $\sigma$  be a bijection between subsets  $J$  and  $K$  of  $S$ , extended to an isomorphism  $\sigma : W_J \rightarrow W_K$ . Define a *twisted*  $J$ -conjugacy class  $\mathfrak{C}_{\sigma} := \mathfrak{C}_{\sigma}^J$

(or  $\mathfrak{C}_{\sigma,w}$  when we want to stress a representative  $w$  of the class) to be a set of the form  $\{\sigma(u)wu^{-1} \mid u \in W_J\}$ . Such twisted  $J$ -conjugacy classes partition the double coset  $\sigma(W_J)wW_J = W_K dW_J$ . Note that when  $\sigma = 1$  and  $J = K$ , this set is a straightforward  $J$ -conjugacy class, and when in addition  $J = S$ , we have a standard conjugacy class. Note also that when  $J = S$ ,  $\sigma$  becomes an automorphism of  $W$ , which is the case studied in [6]. It is worth bearing these special cases in mind as we will make the following definitions in the generality of twisted  $J$ -conjugacy classes.

For  $w, w' \in \mathfrak{C}_\sigma$ , we say  $w \rightarrow_J w'$ , provided that there exists a sequence  $r_1, r_2, \dots, r_m$  of elements of  $J$  and a sequence  $w_0, \dots, w_m$  of elements of  $\mathfrak{C}_\sigma$  such that if  $w_0 = w$  and  $w_i = \sigma(r_i)w_{i-1}r_i$  for  $1 \leq i \leq m$ , then  $w_m = w'$  and  $l(w_i) \leq l(w_{i-1})$  with  $w_i \neq w_{i-1}$  for  $1 \leq i \leq m$ .

Let  $\mathfrak{C}_{\sigma,\min}$  be the set of shortest elements in the twisted  $J$ -conjugacy class  $\mathfrak{C}_\sigma$ . We say that a twisted  $J$ -conjugacy class  $\mathfrak{C}_\sigma$  is *reducible* if for all  $w \in \mathfrak{C}_\sigma$ , there exists  $v \in \mathfrak{C}_{\sigma,\min}$  such that  $w \rightarrow_J v$ .

Let  $w, w' \in \mathfrak{C}_\sigma$  have the same length. We say  $w \sim_{GP} w'$  if there exists a sequence of  $x_i \in W_J$  and  $w_i \in \mathfrak{C}_\sigma$  such that  $w = w_0$ ,  $\sigma(x_i)w_i x_i^{-1} = w_{i+1}$ , and  $w_{n+1} = w'$  with  $l(w_i) = l(w)$ , and either  $l(\sigma(x_i)w_i) = l(x_i) + l(w_i)$  or  $l(w_i x_i^{-1}) = l(w_i) + l(x_i)$  for each  $i = 0, 1, \dots, n$ . The subscript  $GP$  is used as to the best of my knowledge this equivalence was first introduced in [7].

### 3 Reducibility in Twisted $J$ -Conjugacy Classes

In this section, we will provide proofs of Properties I and II. We will actually prove the marginally more general result of allowing our  $J$ -conjugacy classes to be twisted, as this will help smooth the inductive argument.

But first we recall some results from [4].

Let  $d \in \mathcal{D}_{K,J}$  and let  $\sigma$  be a bijection from  $J$  to  $K$ . Let  $K_l \subseteq K$  be the set  $\{s \in K \mid sd = ds' \text{ for some } s' \in J\}$ , and  $J_r$  the subset of  $J$  consisting of those  $s'$  for which  $sd = ds'$  for some  $s \in K_l$ . Let  $\sigma_d$  be the bijection from  $K_l$  to  $J_r$  defined by  $d$  (setting  $\sigma_d(s) = s' \in J_r$  if  $sd = ds'$ ). Denote  $J_l = \sigma^{-1}(K_l)$ .

**Lemma 3.1.** [4, (2.3), (2.4)] *Let  $dw$  be an element of the twisted  $J$ -conjugacy class  $\mathfrak{C}_{\sigma,dw}$  with  $d \in \mathcal{D}_{K,J}$  and  $w \in W_J$ . Then*

- (i)  *$dw$  is a shortest element of  $\mathfrak{C}_{\sigma,dw}$  if and only if  $w$  is a shortest element of its twisted  $J_l$ -conjugacy class  $\mathfrak{C}_{\sigma_d\sigma,w}$ ;*
- (ii)  *$dw$  is reducible in its twisted  $J$ -conjugacy class  $\mathfrak{C}_{\sigma,dw}$  if and only if  $w$  is reducible in its twisted  $J_l$ -conjugacy class  $\mathfrak{C}_{\sigma_d\sigma,w}$ .*

We will also need the following result on full conjugacy classes.

**Theorem 3.2.** [6, Theorem (2.6)] *If  $\mathfrak{C}$  is a twisted conjugacy class of  $W$ , then  $\mathfrak{C}$  is reducible and  $w \sim_{GP} w'$  for any  $w, w' \in \mathfrak{C}_{\min}$ .*

This gives Properties I and II in the case  $J = S$ . It remains to generalize these to arbitrary  $J \subseteq S$ .

**Theorem 3.3.** *All twisted  $J$ -conjugacy classes in a finite Coxeter group  $W$  are reducible.*

*Proof.* Let  $(W, S)$  be a finite Coxeter system and  $J \subseteq S$ . Since the result holds for  $J = S$  by [6], we may assume  $J \subset S$ . We will proceed by induction on  $|J|$ .

If  $|J| = 1$ , the result can easily be seen since any twisted  $J$ -conjugacy class has at most two elements. Suppose inductively that for  $|J| \leq k$ , all twisted  $J$ -conjugacy classes are reducible.

Let  $|J| = k + 1$ . It is clear that any element of  $W_K d W_J$  can be reduced to an element of the form  $dw$ , so it is sufficient to show that given an arbitrary  $dw$  for  $w \in W_J$  and  $d$  distinguished, we can reduce  $dw$  to a shortest element of its twisted  $J$ -conjugacy class. By Lemma 3.1, this is equivalent to showing that  $w$  is reducible in its twisted  $J_l$ -conjugacy class, where the twist is  $\sigma_d \sigma$ .

Now if  $J_l \subsetneq J$ , we have the reducibility of  $w$  by induction. On the other hand, if  $J_l = J$ , since  $w \in W_J$ , we need  $w$  to be reducible in its twisted conjugacy class in  $W = W_J$ . This holds by Theorem 3.2, which completes the proof.  $\square$

**Theorem 3.4.** *For any twisted  $J$ -conjugacy class  $\mathfrak{C}_\sigma$  and  $w, w' \in \mathfrak{C}_{\sigma, \min}$ , we have  $w \sim_{GP} w'$ .*

*Proof.* We need to show the existence of a sequence of twisted conjugations by elements of  $W_J$  connecting  $w$  and  $w'$ . Clearly,  $w \sim_{GP} du$  and  $w' \sim_{GP} du'$  for  $d \in \mathcal{D}_{K, J}$  and some  $u, u' \in W_J$ , so it suffices to show that any pair  $du, du' \in \mathfrak{C}_{\sigma, \min}$  satisfy  $du \sim_{GP} du'$ . To show this, in turn, it is sufficient by Lemma 3.1 to show  $u \sim_{GP} u'$  in their  $\sigma_d \sigma$ -twisted  $J_l$ -conjugacy class within  $W_J$ .

We use induction on  $|J|$ . If  $|J| = 1$ , the result is trivial since each twisted  $J$ -conjugacy class has at most two elements. Suppose it holds when  $|J| \leq k$ .

Let  $|J| = k + 1$ . Take arbitrary elements in  $\mathfrak{C}_{\sigma, w, \min}$  of forms  $du$  and  $du'$ . Then by Lemma 3.1(i), we have that  $u$  and  $u'$  are minimal elements of the twisted  $J_l$ -conjugacy class  $\mathfrak{C}_{\sigma_d \sigma, u}$ .

If  $J_l \subsetneq J$ , then by our induction hypothesis, we have  $u \sim_{GP} u'$ . In other words, we have a sequence of twisted conjugations by elements of  $J_l$  connecting  $u$  and  $u'$ , where the twist is  $\sigma_d \sigma$ . This same sequence then provides the required sequence connecting  $du$  with  $du'$  in the twisted  $J$ -conjugacy class  $\mathfrak{C}_{\sigma, w}$ .

On the other hand, if  $J_l = J$ , then  $u$  and  $u'$  are minimal elements in their  $\sigma_d \sigma$ -twisted  $J$ -conjugacy class in  $W_J$ , i.e., they are actually shortest elements in the full  $\sigma_d \sigma$ -twisted conjugacy class within the finite Coxeter group  $W_J$ . Then by Theorem 3.2,  $u \sim_{GP} u'$  and so there exists a sequence of  $\sigma_d \sigma$ -twisted conjugations by elements of  $W_{J_l}$  connecting them. Exactly,

the same sequence of conjugations, twisting instead by  $\sigma$  only, will connect  $du$  and  $du'$  as required by the definition of  $\sim_{GP}$ .  $\square$

#### 4 Centralizers in Iwahori–Hecke Algebras

Given a set  $\{\xi_s \mid s \in S\}$  of indeterminates and  $R = \mathbb{Z}[\xi_s]_{s \in S}$ , where  $\xi_s = \xi_{s'}$  if  $s$  and  $s'$  are conjugate, the *Iwahori–Hecke algebra*  $\mathcal{H}$  over  $R$  is the associative algebra generated by the set  $\{\tilde{T}_s \mid s \in S\}$  with relations

$$\tilde{T}_s^2 = \tilde{T}_1 + \xi_s \tilde{T}_s$$

and

$$\tilde{T}_w = \tilde{T}_{s_{i_1}} \cdots \tilde{T}_{s_{i_r}}$$

when  $w = s_{i_1} \cdots s_{i_r}$  is a reduced expression. The relationship between this definition of  $\mathcal{H}$  and the more standard one over  $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$  can be found in [4].

We write  $\mathcal{H}^+$  for the set of elements of  $\mathcal{H}$  whose terms  $\tilde{T}_w$  have coefficients in  $R^+ := \mathbb{N}[\xi_s]_{s \in S}$ . That is,  $\mathcal{H}^+$  is the  $R^+$ -span of the set  $\{\tilde{T}_w \mid w \in W\}$ . This positive part  $\mathcal{H}^+$  of  $\mathcal{H}$  is given a partial order by setting for  $a, b \in \mathcal{H}^+$ ,  $a \leq b$  if  $b - a \in \mathcal{H}^+$ . If  $h \in \mathcal{H}^+$ , we say that  $h$  is *primitive* if the monomial coefficients of terms  $\tilde{T}_w$  in  $h$  have no common factors.

If we set  $Z_{\mathcal{H}}(\mathcal{H}_J)^+ = Z_{\mathcal{H}}(\mathcal{H}_J) \cap \mathcal{H}^+$  and define  $Z_{\mathcal{H}}(\mathcal{H}_J)_{\min}^+$  to be the primitive minimal elements of  $Z_{\mathcal{H}}(\mathcal{H}_J)^+$  with respect to the partial order  $\leq$  on  $\mathcal{H}^+$ , then the main theorem in [3] states that  $Z(\mathcal{H})_{\min}^+$  is an  $R$ -basis for the centre of  $\mathcal{H}$ , called the *minimal basis*. The minimal basis consists of *class elements*  $\tilde{\Gamma}_C$  characterized by the properties that  $\tilde{\Gamma}_C|_{\xi=0} = \tilde{T}_C$  (the conjugacy class sum) and  $\tilde{\Gamma}_C - \tilde{T}_C$  contains no shortest elements of any conjugacy class.

To extend this structure to a centralizer of a parabolic subalgebra  $\mathcal{H}_J$  for  $J \subseteq S$ , let  $\mathfrak{C}$  be a  $J$ -conjugacy class, and  $Z_{\mathcal{H}}(\mathcal{H}_J)_{\min}^+$  the set of primitive minimal positive elements of the centralizer  $Z_{\mathcal{H}}(\mathcal{H}_J)$ . Then the generalization requires proving the existence of  $J$ -class elements  $\tilde{\Gamma}_{\mathfrak{C}}$  which specialize to  $J$ -class sums, and that these  $J$ -class elements form the set  $Z_{\mathcal{H}}(\mathcal{H}_J)_{\min}^+$  and an  $R$ -basis for  $Z_{\mathcal{H}}(\mathcal{H}_J)$ .

In [4], the above generalization to the centralizer setting was reduced to proving Properties I and II. This was achieved only in types  $A_n$  and  $B_n$ . Now we complete the generalization of Properties I and II to any  $J \subseteq S$ , i.e., the generalization to centralizers of parabolic subalgebras of Iwahori–Hecke algebras of finite Coxeter groups.

**Theorem 4.1.** *Suppose  $J \subseteq S$ . Then*

$$Z_{\mathcal{H}}(\mathcal{H}_J)_{\min}^+ = \{\tilde{\Gamma}_{\mathfrak{C}} \mid \mathfrak{C} \in \text{ccl}_J(W)\}$$

*and  $Z_{\mathcal{H}}(\mathcal{H}_J)_{\min}^+$  is an  $R$ -basis for  $Z_{\mathcal{H}}(\mathcal{H}_J)$ .*

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