On bases of centres of Iwahori–Hecke algebras of the symmetric group

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Abstract

In 1990, using norms, the second author constructed a basis for the centre of the Hecke algebra of the symmetric group S_n over $\mathbb{Q}[\xi]$ [Trans. Amer. Math. Soc. 317 (1) (1990) 361–392]. An integral "minimal" basis was later given by the first author in [J. Algebra 221 (1) (1999) 1–28], following [M. Geck, R. Rouquier, Centers and simple modules for Iwahori–Hecke algebras, in: Finite Reductive Groups, Luminy, 1994, Birkhäuser, Boston, MA, 1997, pp. 251–272]. In principle one can then write elements of the norm basis as integral linear combinations of minimal basis elements.

In this paper we find an explicit non-recursive expression for the coefficients appearing in these linear combinations. These coefficients are expressed in terms of certain permutation characters of S_n.

In the process of establishing this main theorem, we prove the following items of independent interest: a result on the projection of the norms onto parabolic subalgebras, the existence of an inner product on the Hecke algebra with some interesting properties, and the existence of a partial ordering on the norms.

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0. Introduction

There are now three distinct descriptions of the centre of the Iwahori–Hecke algebra $\mathcal{H}$ of the symmetric group S_n. It has two nice bases, one consisting of norms over $\mathbb{Q}[\xi]$ [13], and one a "minimal basis" of class elements over $\mathbb{Z}[\xi]$ [5,9]. Thirdly, it is now known that the symmetric functions in Murphy elements are precisely the centre of $\mathcal{H}$ over $\mathbb{Z}[\xi]$ [10], and it follows that the elementary symmetric functions in Murphy elements generate the centre over $\mathbb{Z}[\xi]$. A natural question is then to ask "How are these descriptions related?" The relationship between the elementary symmetric functions of Murphy elements and the minimal basis is now known precisely at least in one direction [4], but relationships with the norm basis have been opaque. Furthermore, the elucidation of the connections between the norm basis and the other bases is of interest since the norms of [13] are natural central structures which have been used to define Brauer-type homomorphisms for Hecke algebras [3,6,12] and q-Schur algebras [2].

The goal of this paper is to describe an explicit relationship between the norm basis and the minimal basis for the centre of the Hecke algebra of the symmetric group S_n. This relationship is given by an expression for the coefficients of class elements (the minimal basis) as they appear in the norms. These coefficients are described in terms of the values of certain permutation characters of S_n.

Let α and λ be partitions of n, with w_α an element of the conjugacy class C_α of S_n. Let l_λ and l_α be the lengths of the minimal elements in the corresponding conjugacy classes of S_n, and let ξ be the defining indeterminate of the Hecke algebra. Let $(1_{S_\lambda})^{S_n}$ be the permutation character of S_n which arises from the induction to S_n of the trivial character on the parabolic subgroup S_λ. The main result is as follows.

Theorem 9.2. *Let b_α be an element of the norm basis and let Γ_λ be an element of the minimal basis. Then*

$$b_\alpha = \sum_{\lambda \vdash n} (1_{S_\lambda})^{S_n}(w_\alpha) \xi^{l_\lambda - l_\alpha} \Gamma_\lambda.$$

A considerable amount of machinery, which involves several results of independent interest, is developed in the course of obtaining Theorem 9.2.

The preliminary Section 1 introduces most of the basic definitions and notation used throughout the paper. The reader may wish to skim this section and return for reference as required later in the paper. Section 2 contains results about double coset representatives of parabolic subgroups in the symmetric group S_n which are required for Section 8. A formula for the square of the Hecke algebra element corresponding to a distinguished double coset representative is given in Section 3. In Section 4, the main properties of the bases for the centre are briefly reprised. Section 5 introduces an inner product on the Hecke algebra and gives some elementary properties. In Section 6, we find the coefficient of Γ_λ in b_α when α is trivial (Theorem 6.3), while in Section 7 we determine the coefficient of the Coxeter class element $\Gamma_{(n)}$ in b_α for all $\alpha \vdash n$ (Theorem 7.4). To establish Theorem 7.4, we show that the basis of norms satisfies a partial order consistent with the refinement order on partitions (Theorem 7.2). The descriptions of coefficients in Theorems 6.3 and 7.4 are later

made redundant by Theorem 9.2, but are necessary for its proof. Section 8 gives the main projection theorem (Theorem 8.1), which uses a Mackey-type decomposition to give a rule for projecting norms onto a maximal parabolic subalgebra. This result has been used to study the Brauer homomorphism in [3]. In Section 9, Theorem 8.1 is generalized to a rule for projecting onto arbitrary parabolic subalgebras (Theorem 9.1), and the main theorem quoted above is deduced. Finally, the main result is demonstrated in Section 10 with some examples.

1. Definitions and notation

Throughout we take $\mathbb{N}$ to mean the set of non-negative integers.

1.1. Compositions, partitions and multipartitions

A *composition* λ is a finite ordered set of positive integers. If $\lambda = (\lambda_1, \ldots, \lambda_r)$, the λ_i are called the *components* of λ. If λ is a composition we write $|\lambda| = \sum_{i=1}^{r} \lambda_i$. If $|\lambda| = n$ we say λ is a *composition of n*, and we write $\lambda \vDash n$. Two compositions are said to be *conjugate* if they have the same components.

If $\lambda = (\lambda_1, \ldots, \lambda_r) \vDash n$ then we define $\lambda - 1$ to be the composition of n obtained from λ by replacing each $\lambda_i > 1$ by the juxtaposed ordered pair of positive integers $\lambda_i - 1$ and 1. For example, if $\lambda = (3, 4, 1, 7)$ then $\lambda - 1 = (2, 1, 3, 1, 1, 6, 1)$.

If λ and μ are compositions of n and either $\lambda = \mu$ or λ can be obtained from μ by adding together adjacent components of μ, we say μ is a *refinement* of λ and write $\mu \leqslant \lambda$.

A *partition* of n is a composition whose components are weakly decreasing from left to right. If λ is a partition of n we write $\lambda \vdash n$.

A *multipartition* is a finite ordered set of partitions. A λ-*multipartition of n* for $\lambda = (\lambda_1, \ldots, \lambda_r) \vDash n$ is an ordered set of partitions $\theta = (\theta_1, \ldots, \theta_r)$ with $\theta_i \vdash \lambda_i$ for each $i = 1, \ldots, r$. Note that from any multipartition θ of n we can derive a unique composition λ of n by removing the internal parentheses. We call this unique composition λ, the *derived composition* of the multipartition θ. For example, $\theta = ((4, 1), (3, 2, 1), (2, 1))$ has derived composition $\lambda = (4, 1, 3, 2, 1, 2, 1)$ of $n = 14$.

By the *components* of a λ-multipartition of n, we mean the components of its constituent partitions. If $\alpha \vdash n$, then a λ-*multipartition of α* is a λ-multipartition of n whose components are the components of α. Let Λ_λ be the set of λ-multipartitions of n, and let $\Lambda_\lambda(\alpha)$ be the set of λ-multipartitions of α. For example, a $(3, 5, 2)$-multipartition of $(3, 2, 2, 1, 1, 1)$ is $((2, 1), (3, 1, 1), (2))$.

Note that for many choices of λ and α there are no λ-multipartitions of α; for instance there are no $(3, 2)$-multipartitions of $(4, 1)$.

1.2. The symmetric group

Let S_n be the symmetric group on n letters with generating set of simple reflections

$$S := \big\{ s_i = (i \ \ i + 1) \mid 1 \leqslant i \leqslant n - 1 \big\}.$$

We use both the s_i notation and the cycle notation as expedient. We adopt the convention that $S_0 = S_1 = \{(1)\}$.

We say an expression for $w \in S_n$ is *reduced* if there is no way to write w as a word in fewer generators. In this case we say the length $\ell(w)$ of w is this minimal number of generators. Symmetric groups act on sets of vectors in Euclidean space known as *root systems*. One can define the concepts of *positive* and *negative* roots which in turn can be used to describe the length of an element of S_n. In particular if Φ^+ and $\Phi^- := \{-v \mid v \in \Phi^+\}$ are the sets of positive and negative roots respectively, then $\ell(w) = |w(\Phi^+) \cap \Phi^-|$. A set of positive roots for the root system of S_n is the set $\Phi^+ = \{e_i - e_j \mid 1 \leqslant i < j \leqslant n\}$ where $\{e_i \mid 1 \leqslant i \leqslant n\}$ is the set of standard basis vectors for $\mathbb{R}^n$.

For $\lambda = (\lambda_1, \ldots, \lambda_k) \vDash n$ define

$$S_\lambda := S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k},$$

where for $\lambda_i > 1$, S_{λ_i} is the subgroup of S_n generated by the set $\{s_{\lambda_1+\cdots+\lambda_{i-1}+1}, \ldots, s_{\lambda_1+\cdots+\lambda_i-1}\}$, and for $\lambda_i = 1$, S_{λ_i} is the trivial subgroup. Such a subgroup S_λ is called a *parabolic* subgroup of S_n. Note that $S_\mu \leqslant S_\lambda$ if and only if $\mu \leqslant \lambda$ (that is, μ is a refinement of λ).

If $\theta = (\theta_1, \ldots, \theta_t)$ is a multipartition (the θ_i are partitions), then set $S_\theta := S_{\theta_1} \times \cdots \times S_{\theta_t}$.

The unique element of a S_λ–S_μ double coset of S_n of minimal length is called a *distinguished* double coset representative (such elements are well known to be unique—see [1]). Let $\mathcal{D}_{\lambda\mu}$ denote the set of distinguished S_λ–S_μ double coset representatives in S_n.

The conjugacy classes of S_n are indexed by partitions λ of n. Write C_λ for the conjugacy class consisting of elements of S_n of cycle type λ. Write $\underline{C_\lambda}$ for the sum of elements in the conjugacy class C_λ. If $\lambda = (\lambda_1, \ldots, \lambda_r)$ then set

$$w_\lambda = (s_1 \ldots s_{\lambda_1-1})(s_{\lambda_1+1} \ldots s_{\lambda_1+\lambda_2-1}) \ldots (s_{\lambda_1+\cdots+\lambda_{r-1}+1} \ldots s_{\lambda_1+\cdots+\lambda_r-1})$$

where we take each empty sequence of s_i's (when $\lambda_j = 1$) to be the identity. Then w_λ is a *Coxeter element* of the subgroup S_λ, and also a minimal length element of the conjugacy class C_λ in S_n.

In S_λ for $\lambda \vDash n$, the conjugacy classes are indexed by the set Λ_λ of λ-multipartitions of n. In particular, in $S_{(k,n-k)}$, the classes are indexed by $(k, n-k)$-multipartitions of n. If $\lambda \vDash n$ and $\theta \in \Lambda_\lambda$, then C_θ denotes the conjugacy class in S_λ corresponding to the composition of n derived from θ.

As usual, $|C_\theta|_{S_\lambda}$ denotes the size of the conjugacy class C_θ in S_λ.

Let l_λ be the length of a shortest element of the conjugacy class C_λ, that is, $l_\lambda = \ell(w_\lambda)$.

For $w \in S_n$ and fixed $k \in \{1, \ldots, n-1\}$, define $\#(w)$ to be the minimal number of times the generator $s_k = (k\ k+1)$ must appear in any reduced expression for w. Unless otherwise noted, k is assumed to be fixed throughout this paper. For $w \in S_n$ and $I \subseteq \{1, \ldots, n\}$, we write $w.I$ for the image of the action of w on the set I.

The *Bruhat order* on S_n is defined as follows. For $v, w \in S_n$, we say $v \leqslant w$ if there exists a reduced expression of v which is a subword of a reduced expression for w.

For any groups $H \leqslant G$, we use the standard notation $C_G(H)$ and $N_G(H)$ to indicate the *centralizer* and *normalizer* respectively, of H in G.

1.3. The Hecke algebra

In this paper we use the normalized version of the generators for the Hecke algebra, giving us an algebra over the ring $\mathbb{Z}[\xi]$, where ξ is an indeterminate. The exact connection between this definition and the standard definition over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ is given in the remark below.

The Iwahori–Hecke algebra $\mathcal{H} := \mathcal{H}_n$ of S_n is the associative $\mathbb{Z}[\xi]$-algebra generated by the set $\{\tilde{T}_s \mid s \in S\}$ with identity $\tilde{T}_1$ and subject to the relations

$$\tilde{T}_{s_i} \tilde{T}_{s_j} = \tilde{T}_{s_j} \tilde{T}_{s_i} \quad \text{if } |i - j| \geqslant 2,$$

$$\tilde{T}_{s_i} \tilde{T}_{s_{i+1}} \tilde{T}_{s_i} = \tilde{T}_{s_{i+1}} \tilde{T}_{s_i} \tilde{T}_{s_{i+1}} \quad \text{for } 1 \leqslant i \leqslant n - 2,$$

$$\tilde{T}_{s_i}^2 = \tilde{T}_1 + \xi \tilde{T}_{s_i} \quad \text{for } 1 \leqslant i \leqslant n - 1.$$

If $w = s_{i_1} \dots s_{i_r}$ is a reduced expression for w, then we write $\tilde{T}_w := \tilde{T}_{s_{i_1}} \dots \tilde{T}_{s_{i_r}}$. Then $\mathcal{H}$ is a free $\mathbb{Z}[\xi]$-module with basis $\{\tilde{T}_w \mid w \in S_n\}$.

When specialization of $\mathcal{H}$ to $\xi = 0$ is used in this paper it will be assumed that the specialization is to the group algebra $\mathbb{Z}S_n$. If $h \in \mathcal{H}$ we write $h|_{\xi=0}$ for the specialization of h at $\xi = 0$.

If $\lambda \vDash n$, we let $\mathcal{H}_\lambda$ denote the parabolic subalgebra of $\mathcal{H}_n$ generated by $\{\tilde{T}_s \mid s \in S'\}$, where S' is the subset of S consisting of the simple reflections which generate the parabolic subgroup S_λ. For any multipartition θ of n with derived composition λ of n, we define $\mathcal{H}_\theta := \mathcal{H}_\lambda$.

Remark. Set $\tilde{T}_s := q^{-1/2} T_s$ for $s \in S$ and $\xi = q^{1/2} - q^{-1/2}$. Then $\mathcal{H}$ is a subalgebra of $\mathcal{H}_q$, the more standard Hecke algebra generated by $\{T_s \mid s \in S\}$ over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$. A principal reason for defining the algebra with normalized generators is that doing so gives $\mathcal{H}$ a natural positivity and an associated partial order on the positive cone. Many results on the centre of $\mathcal{H}$ have more natural statements and proofs when the algebra is defined in this way. The main results of this paper are all readily translated back to statements over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$.

Let $\mathcal{H}^+ = \sum_{w \in S_n} \mathbb{N}[\xi] \tilde{T}_w$. For $a, b \in \mathcal{H}^+$ we say $a \leqslant b$ when $b - a \in \mathcal{H}^+$. If in addition $b - a \neq 0$ then we write $a < b$. The partial order restricts to the positive cone of the base ring, $\mathbb{N}[\xi] = \mathbb{Z}[\xi]^+$. For $a, b \in \mathbb{N}[\xi]$, we say $a \leqslant b$ when $b - a \in \mathbb{N}[\xi]$, and $a < b$ if in addition $b - a \neq 0$.

If A is a subalgebra of $\mathcal{H}$, then the *centralizer of A in $\mathcal{H}$*, denoted $Z_\mathcal{H}(A)$, is the set of elements in $\mathcal{H}$ which commute with every element of A. The *centre* of $\mathcal{H}$ is $Z(\mathcal{H}) := Z_\mathcal{H}(\mathcal{H})$. Set $Z(\mathcal{H})^+ := Z(\mathcal{H}) \cap \mathcal{H}^+$.

We say that an element $\tilde{T}_w$ in $\mathcal{H}_n$ *contains* a particular generator $s \in S$ when $s \leqslant w$ in the Bruhat order. We also say (with some abuse of language) that $h = \sum_{w \in S_n} r_w \tilde{T}_w \in \mathcal{H}_n$ *contains* $\tilde{T}_w$, or that $\tilde{T}_w$ *occurs* in h, if $r_w \neq 0$.

2. Double cosets of maximal parabolic subgroups

This section and the next contain a number of results needed for later sections, most of which appear in [12] but not, as far as we can see, in the available literature.

Recall that $k \in \{1, \ldots, n-1\}$ is fixed. Also, throughout this section we let $M = \min\{k, n-k\}$ and $\lambda = (k, n-k) \vDash n$. For $0 \leqslant m \leqslant M$ and $w \in S_n$, set

$$d_m := (k\ k+m)(k-1\ k+m-1)\ldots(k+1-m\ k+1),$$

$$D := \{d_m \mid 0 \leqslant m \leqslant M\},$$

$$\|w\| := \big|w.\{1, \ldots, k\} \cap \{k+1, \ldots, n\}\big|.$$

Proposition 2.1.

(1) *The elements d_m satisfy the following*:
   (a) $d_m^{-1} = d_m$,
   (b) $\ell(d_m) = m^2$,
   (c) $\#(d_m) = m = \|d_m\|$.
(2) $D = \mathcal{D}_{\lambda\lambda}$.

Proof. Part (1)(a) is obvious since d_m is a product of disjoint transpositions.

The results of (1)(b) and (1)(c) are trivial for $m = 0$ and $m = 1$, so we assume $m \geqslant 2$. Using a standard root system argument (see for example [1]), observe that d_m takes the set $\{e_{k-i} - e_{k+1+j} \mid 0 \leqslant i, j \leqslant m-1\}$ of m^2 positive roots to negative roots, which implies that $\ell(d_m) \geqslant m^2$. Since $d_m = d_{m-1}(s_{k-m+1} \ldots s_{k-1} s_{k+m} \ldots s_k)$, we have that $\ell(d_m) \leqslant \ell(d_{m-1}) + 2m - 1$. By induction, $\ell(d_{m-1}) = (m-1)^2$. Hence $\ell(d_m) \leqslant (m-1)^2 + 2m - 1 = m^2$, giving $\ell(d_m) = m^2$, which proves part (1)(b).

As above, write $d_m = d_{m-1}ws_k$ where $\#(w) = 0$. By induction assume that $\#(d_{m-1}) = m - 1$. Therefore, $\#(d_m) \leqslant m$. From the definition of d_m it is immediate that $\|d_m\| = m$. Then, since $\|d_m\| \leqslant \#(d_m)$, we conclude that $\#(d_m) = m = \|d_m\|$ as required for (1)(c).

Note that if w and u are in the same S_λ–S_λ double coset of S_n, then $\|w\| = \|u\|$. As $\|d_m\| = m$ for each m, we have that each d_m lies in a distinct S_λ–S_λ double coset. Observe that $|D| = M + 1$, and since there are $M + 1$ distinct S_λ–S_λ double cosets [11, Theorem 1.3.10], D is a set of double coset representatives. Since any element w of $S_\lambda d_m S_\lambda$ takes the m^2 above-mentioned positive roots to negative roots, it follows that $\ell(w) \geqslant m^2$. Thus d_m is distinguished, proving (2). $\square$

Corollary 2.2. *Let $x \in S_n$. Then*

(1) $\#(x) = \|x\|$,
(2) $x \in S_\lambda d_m S_\lambda$ *if and only if* $\#(x) = m$.

Proof. Suppose that $x \in S_\lambda d_m S_\lambda$. Then we can write $x = y d_m z$ with $\#(y) = 0 = \#(z)$, and clearly, $\#(y d_m z) \leqslant \#(d_m)$. Recall that $\#(d_m) = \|d_m\|$ from Proposition 2.1(1)(c). Since x and d_m are both elements of $S_\lambda d_m S_\lambda$, we have $\|x\| = \|d_m\|$. Hence, $\#(x) = \#(y d_m z) \leqslant$

$\#(d_m) = \|d_m\| = \|x\|$. Note that each occurrence of s_k in any expression for x produces at most one element of $x.\{1,\ldots,k\} \cap \{k+1,\ldots,n\}$. Thus, $\#(x) \geqslant \|x\|$, which proves (1).

We use part (1) and arguments in its proof to establish (2). For $x \in S_\lambda d_m S_\lambda$, we have $\#(x) = \|x\| = \|d_m\| = m$. Conversely, if $\#(x) = m$, then $\|x\| = m = \|d_m\|$, which implies that $x \in S_\lambda d_m S_\lambda$, and the proof is complete. $\quad\square$

For $0 \leqslant m \leqslant M$, let

$$P_m := S_\lambda \cap S_\lambda^{d_m}.$$

Proposition 2.3.

(1) $P_m = P_m^{d_m}$.
(2) $P_m = S_{(k-m,m,m,n-k-m)}$.

Proof. Part (1) is immediate from Proposition 2.1(1)(a).

Clearly $S_{(k-m,m,m,n-k-m)} \leqslant S_\lambda$. Observe that d_m normalizes $S_{(k-m,m,m,n-k-m)}$, and so

$$S_{(k-m,m,m,n-k-m)} = S_{(k-m,m,m,n-k-m)}^{d_m} \leqslant S_\lambda^{d_m},$$

giving $S_{(k-m,m,m,n-k-m)} \leqslant P_m$.

If $s_{k-m} \in S_\lambda^{d_m}$, then $d_m s_{k-m} d_m \in S_\lambda$. But

$$d_m s_{k-m} d_m = s_{k-m} s_{k-m+1} \ldots s_{k-1} s_k s_{k-1} \ldots s_{k-m+1} s_{k-m}$$

which implies that $s_k \in S_\lambda$, a contradiction. Hence, $s_{k-m} \notin S_\lambda^{d_m}$. Similarly, $s_{k+m} \notin S_\lambda^{d_m}$. Since P_m is parabolic it follows that $P_m \leqslant S_{(k-m,m,m,n-k-m)}$, and (2) is proved. $\quad\square$

Corollary 2.4. *If $w \in P_m$ then $\ell(d_m w d_m) = l(w)$.*

Proof. Immediate from Proposition 2.3(2) and the definition of d_m. $\quad\square$

Corollary 2.5. *Let U be a parabolic subgroup of P_m. Let r be a distinguished right coset representative for U in P_m. Then $d_m r d_m$ is a distinguished right coset representative for U^{d_m} in P_m.*

Proof. This is immediate from Corollary 2.4. $\quad\square$

3. The square of the Hecke algebra element corresponding to a distinguished double coset representative

The goal of this section is to prove Proposition 3.5, which gives an expansion for the square of the Hecke algebra element corresponding to a distinguished double coset representative of a maximal parabolic subgroup. Proposition 3.5 forms part of the machinery

needed for our analysis in Section 8 of the projection of the norm basis onto a maximal parabolic subalgebra. Throughout this section we let $\lambda = (k, n - k) \vDash n$.

For $x, y, z \in S_n$, define the polynomial $f_{xyz} \in \mathbb{N}[\xi]$ to be the coefficient of $\tilde{T}_z$ occurring in the expansion of $\tilde{T}_x \tilde{T}_y$. That is, write

$$\tilde{T}_x \tilde{T}_y = \sum_{z \in S_n} f_{xyz} \tilde{T}_z. \tag{3.1}$$

Lemma 3.1. *Let* $x, y, z \in S_n$, *with* $f_{xyz} \neq 0$.

(1) $\#(x) - 1 \leqslant \#(s_k x) \leqslant \#(x) + 1$.
(2) *If* $\#(x) \neq \#(y)$ *then* $\#(z) \geqslant 1$.

Proof. Firstly note that (2) follows immediately from (1) by induction on the length of x or y. For (1), the upper bound is clear.

We will use induction on $\#(x) = t$ to establish the lower bound in (1). Write $x = w_1 d_t w_2$, with $w_1, w_2 \in S_\lambda$ (so that $\#(w_1) = \#(w_2) = 0$), and assume that $\#(s_k w_1 d_t w_2) < t - 1$. Then $s_k w_1 d_t w_2 \in S_\lambda d_v S_\lambda$ for some $v < t - 1$. That is, $s_k w_1 d_t w_2 = \hat{x}$ for some $\hat{x} \in S_\lambda d_v S_\lambda$. Then $d_t = w_1^{-1} s_k \hat{x} w_2^{-1}$. By induction, $\#(\hat{x}) - 1 \leqslant \#(s_k \hat{x}) \leqslant \#(\hat{x}) + 1$. In other words,

$$v - 1 \leqslant \#\left(s_k \hat{x}\right) \leqslant v + 1.$$

Since $w_1^{-1}, w_2^{-1} \in S_\lambda$ we have that d_t is an element of either $S_\lambda d_{v-1} S_\lambda$, $S_\lambda d_v S_\lambda$ or $S_\lambda d_{v+1} S_\lambda$, each of which is impossible since $v < t - 1$. $\quad\square$

We recall a result of Shi [14].

Theorem 3.2 [14, Theorem 8]. *Let* $x, y, z \in S_n$. *If* $f_{xyz} \neq 0$ *then* $xy \leqslant z$ *in the Bruhat order.*

An elementary consequence is the following:

Corollary 3.3. *Let* $x, y, z \in S_n$. *If* $f_{xyz} \neq 0$, *then*:

(1) $\#(xy) \leqslant \#(z)$;
(2) *If* $\#(x) = t$ *and* $\#(y) = 0$ *then* $\#(z) = t$.

Lemma 3.4. *For* $j_1 \leqslant j_2$,

$$\tilde{T}_{s_{j_1} \dots s_{j_2}} \tilde{T}_{s_{j_2} \dots s_{j_1}} = \tilde{T}_1 + \xi \sum_{i=j_1}^{j_2} \tilde{T}_{s_{j_1} \dots s_i \dots s_{j_1}}.$$

Proof. Elementary (by induction on j_2). $\quad\square$

Proposition 3.5.

$$\left(\tilde{T}_{d_m}\right)^2 = \tilde{T}_1 + \sum_{w \in S_n} f_w \tilde{T}_w$$

where $\#(w) \geqslant 1$ for $0 \neq f_w \in \mathbb{N}[\xi]$. (Note that we abbreviate $f_{d_m d_m w}$ in the expansion of (3.1) to f_w.)

Proof. For $m = 1$, the proposition is easily verified. Assume $m \geqslant 2$, and write $d_m = d_{m-1} u v s_k$ where $u = s_{k-m+1} s_{k-m+2} \ldots s_{k-1}$ and $v = s_{k+m-1} s_{k+m-2} \ldots s_{k+1}$. Then, since $d_m^{-1} = d_m$ and $uv = vu$, we have

$$\left(\tilde{T}_{d_m}\right)^2 = \tilde{T}_{d_{m-1}} \tilde{T}_u \tilde{T}_v \tilde{T}_{s_k} \tilde{T}_{s_k} \tilde{T}_{v^{-1}} \tilde{T}_{u^{-1}} \tilde{T}_{d_{m-1}}$$

$$= \tilde{T}_{d_{m-1}} \left(\tilde{T}_u \tilde{T}_{u^{-1}} \tilde{T}_v \tilde{T}_{v^{-1}}\right) \tilde{T}_{d_{m-1}} + \xi \tilde{T}_{d_m} \tilde{T}_{v^{-1} u^{-1} d_{m-1}}.$$

Now, $v^{-1} u^{-1} d_{m-1} \in S_\lambda d_{m-1} S_\lambda$ and so by Corollary 2.2(2), $\#(v^{-1} u^{-1} d_{m-1}) = m - 1$. Since $\#(d_m) = \#(d_{m-1}) + 1$, Corollary 3.3 implies that every term of the product $\tilde{T}_{d_m} \tilde{T}_{v^{-1} u^{-1} d_{m-1}}$ contains s_k. Now

$$\tilde{T}_u \tilde{T}_{u^{-1}} = \tilde{T}_1 + \xi \sum_{i=k-m+1}^{k-1} \tilde{T}_{s_{k-m+1} \ldots s_i \ldots s_{k-m+1}}$$

by Lemma 3.4. Similarly, $\tilde{T}_v \tilde{T}_{v^{-1}} = \tilde{T}_1 + \xi \sum_{i=2}^{m} \tilde{T}_{s_{k+i-1} \ldots s_{k+m-1} \ldots s_{k+i-1}}$. Hence

$$\tilde{T}_{d_{m-1}} \left(\tilde{T}_u \tilde{T}_{u^{-1}} \tilde{T}_v \tilde{T}_{v^{-1}}\right) \tilde{T}_{d_{m-1}}$$

$$= \tilde{T}_{d_{m-1}} \left(\tilde{T}_1 + \xi \sum_{i=k-m+1}^{k-1} \tilde{T}_{s_{k-m+1} \ldots s_i \ldots s_{k-m+1}}\right)$$

$$\times \left(\tilde{T}_1 + \xi \sum_{i=2}^{m} \tilde{T}_{s_{k+i-1} \ldots s_{k+m-1} \ldots s_{k+i-1}}\right) \tilde{T}_{d_{m-1}}.$$

By induction, assume

$$\left(\tilde{T}_{d_{m-1}}\right)^2 = \tilde{T}_1 + \sum_{w \in S_n} f'_w \tilde{T}_w$$

where $\#(w) \geqslant 1$ for $f'_w \neq 0$, and again we have abbreviated the coefficient $f_{d_{m-1} d_{m-1} w}$ from (3.1) to f'_w.

The proposition will be proved if we can show that all terms in each of the products

$$\tilde{T}_{d_{m-1}} \tilde{T}_{s_{k-m+1} \ldots s_{k-m+a-1} \ldots s_{k-m+1}} \tilde{T}_{d_{m-1}}, \qquad \tilde{T}_{d_{m-1}} \tilde{T}_{s_{k+b-1} \ldots s_{k+m-1} \ldots s_{k+b-1}} \tilde{T}_{d_{m-1}}$$

and

$$\tilde{T}_{d_{m-1}} \tilde{T}_{s_{k-m+1}\dots s_{k-m+a-1}\dots s_{k-m+1}} \tilde{T}_{s_{k+b-1}\dots s_{k+m-1}\dots s_{k+b-1}} \tilde{T}_{d_{m-1}},$$

contain s_k, for $2 \leqslant a, b \leqslant m$.

The product $(s_{k-m+1}\dots s_{k-m+a-1}\dots s_{k-m+1})(s_{k+b-1}\dots s_{k+m-1}\dots s_{k+b-1})$ adds in length, and since it is an element of S_λ for all a and b, its product with d_{m-1} is length-additive. Hence,

$$\tilde{T}_{s_{k-m+1}\dots s_{k-m+a-1}\dots s_{k-m+1}} \tilde{T}_{s_{k+b-1}\dots s_{k+m-1}\dots s_{k+b-1}} \tilde{T}_{d_{m-1}}$$
$$= \tilde{T}_{s_{k-m+1}\dots s_{k-m+a-1}\dots s_{k-m+1} s_{k+b-1}\dots s_{k+m-1}\dots s_{k+b-1} d_{m-1}}.$$

It is easily determined that

$$d_{m-1} s_{k-m+1}\dots s_{k-m+a-1}\dots s_{k-m+1} d_{m-1} \in S_\lambda d_1 S_\lambda,$$
$$d_{m-1} s_{k+b-1}\dots s_{k+m-1}\dots s_{k+b-1} d_{m-1} \in S_\lambda d_1 S_\lambda$$

and

$$d_{m-1}(s_{k-m+1}\dots s_{k-m+a-1}\dots s_{k-m+1})(s_{k+b-1}\dots s_{k+m-1}\dots s_{k+b-1})d_{m-1} \in S_\lambda d_2 S_\lambda$$

for any a and b with $m \geqslant 2$. That is, by Corollary 2.2(2)

$$\#(d_{m-1} s_{k-m+1}\dots s_{k-m+a-1}\dots s_{k-m+1} d_{m-1})$$
$$= 1 = \#(d_{m-1} s_{k+b-1}\dots s_{k+m-1}\dots s_{k+b-1} d_{m-1})$$

and

$$\#\big(d_{m-1}(s_{k-m+1}\dots s_{k-m+a-1}\dots s_{k-m+1})(s_{k+b-1}\dots s_{k+m-1}\dots s_{k+b-1})d_{m-1}\big) = 2.$$

Thus an application of Corollary 3.3(1) to each of the products

$$\tilde{T}_{d_{m-1}} \tilde{T}_{s_{k-m+1}\dots s_{k-m+a-1}\dots s_{k-m+1} d_{m-1}}, \qquad \tilde{T}_{d_{m-1}} \tilde{T}_{s_{k+b-1}\dots s_{k+m-1}\dots s_{k+b-1} d_{m-1}}$$

and

$$\tilde{T}_{d_{m-1}} \tilde{T}_{(s_{k-m+1}\dots s_{k-m+a-1}\dots s_{k-m+1})(s_{k+b-1}\dots s_{k+m-1}\dots s_{k+b-1})d_{m-1}}$$

completes the proof. □

Corollary 3.6. *Let $v \in P_m$. Then*

$$\tilde{T}_{d_m} \tilde{T}_v \tilde{T}_{d_m} = \tilde{T}_{d_m v d_m} + \sum_{w \in S_n} g_w \tilde{T}_w$$

where $\#(w) \geqslant 1$ for $0 \neq g_w \in \mathbb{N}[\xi]$.

Proof. Since $v \in S_\lambda$, we have $\ell(v d_m) = \ell(v) + \ell(d_m)$. From Proposition 2.3(1), we have that $\hat{v} = d_m v d_m \in P_m$, so also $\ell(d_m \hat{v}) = \ell(d_m) + \ell(\hat{v})$. Thus

$$\begin{aligned}
\tilde{T}_{d_m} \tilde{T}_v \tilde{T}_{d_m} &= \tilde{T}_{d_m} \tilde{T}_{d_m} \tilde{T}_{\hat{v}} \\
&= \left(\tilde{T}_1 + \sum_{u \in S_n} f_u \tilde{T}_u \right) \tilde{T}_{\hat{v}} \quad \text{(where } \#(u) \geqslant 1 \text{ for } f_u \neq 0, \text{ by Proposition 3.5)} \\
&= \tilde{T}_{\hat{v}} + \sum_{u \in S_n} f_u \tilde{T}_u \tilde{T}_{\hat{v}} = \tilde{T}_{d_m v d_m} + \sum_{w \in S_n} g_w \tilde{T}_w
\end{aligned}$$

where $\#(w) \geqslant 1$ when $g_w \neq 0$, by Corollary 3.3(2) since $\#(\hat{v}) = 0$. $\square$

4. Bases for the centre of the Hecke algebra

In this section we introduce the two bases for the centre of the Hecke algebra whose relationship is the main topic of this paper. With the exception of Proposition 4.2, the results in this section appear in the existing literature exactly as stated here, or in slightly less generality.

Some results from [13] have been restated in the context of the Hecke algebra over $\mathbb{Z}[\xi]$, and in the generality of compositions rather than partitions where appropriate.

4.1. The norm basis

Definition 4.1. For $h \in \mathcal{H}$, $\lambda, \mu, \alpha \vDash n$ and $\mu \leqslant \lambda$, we define the *relative norm* of h from S_μ to S_λ to be

$$\mathcal{N}_{S_\lambda, S_\mu}(h) := \sum_{d \in \mathcal{D}} \tilde{T}_{d^{-1}} h \tilde{T}_d,$$

where $\mathcal{D}$ is the set of distinguished right coset representatives of S_μ in S_λ. In addition, define

$$\eta_\lambda := \mathcal{N}_{S_{\lambda-1}, 1}\left(\tilde{T}_{w_\lambda} \right)$$

and

$$b_\alpha := \mathcal{N}_{S_n, S_\alpha}(\eta_\alpha).$$

For any multipartition θ with derived composition λ, we define $\eta_\theta := \eta_\lambda$.

As $C_{(n)}$ is the Coxeter class of S_n, we call $\eta_{(n)}$ the Coxeter class element of $\mathcal{H}_n$. Similarly, η_λ is the Coxeter class element of $\mathcal{H}_\lambda$.

Proposition 4.2. *If α and β are conjugate compositions, then $b_\alpha = b_\beta$.*

Proof. Since any pair of conjugate compositions can be obtained from one another via a sequence of exchanges of adjacent components, it suffices to consider two conjugate compositions which differ by a single adjacent pair. That is, we may assume $\alpha = (\lambda_1, \ldots, \lambda_k, \lambda_{k+1}, \ldots, \lambda_r)$ and $\beta = (\lambda_1, \ldots, \lambda_{k+1}, \lambda_k, \ldots, \lambda_r)$. Then

$$b_\alpha = \mathcal{N}_{S_n, S_\alpha}(\eta_\alpha) = \mathcal{N}_{S_n, S_{(\lambda_1, \ldots, \lambda_k + \lambda_{k+1}, \ldots, \lambda_r)}}\big(\mathcal{N}_{S_{(\lambda_1, \ldots, \lambda_k + \lambda_{k+1}, \ldots, \lambda_r)}, S_\alpha}(\eta_\alpha)\big),$$

with

$$\mathcal{N}_{S_{(\lambda_1, \ldots, \lambda_k + \lambda_{k+1}, \ldots, \lambda_r)}, S_\alpha}(\eta_\alpha)$$
$$= \eta_{(\lambda_1, \ldots, \lambda_{k-1})} \mathcal{N}_{S_{\lambda_k + \lambda_{k+1}}, S_{(\lambda_k, \lambda_{k+1})}}(\eta_{(\lambda_k, \lambda_{k+1})}) \eta_{(\lambda_{k+2}, \ldots, \lambda_r)}.$$

So to show $b_\alpha = b_\beta$ it suffices to show

$$\mathcal{N}_{S_{\lambda_k + \lambda_{k+1}}, S_{(\lambda_k, \lambda_{k+1})}}(\eta_{(\lambda_k, \lambda_{k+1})}) = \mathcal{N}_{S_{\lambda_k + \lambda_{k+1}}, S_{(\lambda_{k+1}, \lambda_k)}}(\eta_{(\lambda_{k+1}, \lambda_k)}). \tag{4.1}$$

These norms are both central in $\mathcal{H}_{\lambda_k + \lambda_{k+1}}$ by [13, Proposition 2.13], and they are images of each other under the algebra automorphism of $\mathcal{H}_{\lambda_k + \lambda_{k+1}}$ defined by reflecting the Dynkin diagram about its midpoint. This automorphism fixes central elements and hence (4.1) holds, proving the lemma. $\square$

Note that if α and β are conjugate compositions of n, there exists $x \in S_n$ such that $S_\alpha^x = S_\beta$. Therefore, we write $\alpha^x = \beta$ in this situation. As before, for any multipartition θ of n with derived composition λ of n, we define $\theta^x := \lambda^x$.

Theorem 4.3 [13]. *Let $\lambda \vDash n$.*

(1) $\eta_\lambda \in Z(\mathcal{H}_\lambda)$.
(2) *Every term with non-zero coefficient in the element $\eta_{(n)}$ contains every simple reflection in its reduced form.*
(3) $\eta_{(n)}$ *contains terms of length $n - 1$ and greater, and specializes on $\xi = 0$ to the Coxeter class sum in $\mathbb{Z}S_n$.*

Proof. Part (1) is [13, Lemma (3.23)(3)]. Part (2) is [13, Lemma (3.14)]. Part (3) is [13, Corollary (3.15) and Lemma (3.23)(4)]. $\square$

Theorem 4.4 [13]. *Let $\lambda \vDash n$.*

(1) *(Transitivity) If $S_\mu \leqslant S_\lambda$ then $\mathcal{N}_{S_n, S_\mu}(h) = \mathcal{N}_{S_n, S_\lambda}(\mathcal{N}_{S_\lambda, S_\mu}(h))$.*

(2) *If $h \in Z(\mathcal{H}_\lambda)$ then $\mathcal{N}_{S_n, S_\lambda}(h) \in Z(\mathcal{H}_n)$.*
(3) *The set $B = \{b_\alpha \mid \alpha \vdash n\}$ is a $\mathbb{Q}[\xi]$-basis for $Z(\mathcal{H})$.*

Proof. Part (1) is [13, Lemma (2.12)]. Part (2) is [13, Proposition (2.13)]. Part (3) is [13, Theorem (3.33)]. □

Note that the set B in Theorem 4.4(3) is indexed by partitions of n (not compositions). See Proposition 4.2.

Theorem 4.5 [13, (2.30)]. *Let $\lambda, \mu \vDash n$. Let S_λ and S_μ be parabolic subgroups of S_n. If $b \in Z_\mathcal{H}(\mathcal{H}_\lambda)$ then*

$$\mathcal{N}_{S_n, S_\lambda}(b) = \sum_{d \in \mathcal{D}_{\lambda\mu}} \mathcal{N}_{S_\mu, S_\lambda^d \cap S_\mu}(\tilde{T}_{d^{-1}} b \tilde{T}_d).$$

Proposition 4.6 [13, (2.32)]. *Let λ and μ be compositions satisfying $|\lambda| + |\mu| = n$, and let $\lambda' \leqslant \lambda$, $\mu' \leqslant \mu$. Let $x \in S_{\lambda'}$ and $y \in S_{\mu'}$. Then*

$$\mathcal{N}_{S_\lambda \times S_\mu, S_{\lambda'} \times S_{\mu'}}(\tilde{T}_{xy}) = \mathcal{N}_{S_\lambda, S_{\lambda'}}(\tilde{T}_x) \mathcal{N}_{S_\mu, S_{\mu'}}(\tilde{T}_y).$$

Proposition 4.7 [13, (3.29)]. *For $\alpha \vdash n$, we have*

$$b_\alpha|_{\xi=0} = [N_{S_n}(S_\alpha) : S_\alpha] \underline{C_\alpha}.$$

4.2. The minimal basis

The minimal basis is the analogue of the class sum basis for the centre of the group algebra. Its existence was shown in [9], and it was explicitly described in [5].

Theorem 4.8 [9]. *There exists a set of elements $\{\Gamma_\lambda \mid \lambda \vdash n\} \subseteq Z(\mathcal{H})$ characterized by the properties*

(1) *$\Gamma_\lambda|_{\xi=0} = \sum_{w \in C_\lambda} \tilde{T}_w$, and*
(2) *$\Gamma_\lambda - \sum_{w \in C_\lambda} \tilde{T}_w$ contains no shortest elements of any conjugacy class.*

These elements form a $\mathbb{Z}[\xi]$-basis for $Z(\mathcal{H})$.

An element in $\mathcal{H}^+$ is said to be *primitive* if, when written as a linear combination of $\{\tilde{T}_w \mid w \in S_n\}$, its coefficients have no common factors over $\mathbb{Z}[\xi]$. The main result of [5] is the following (using the partial order introduced in Section 1.3):

Theorem 4.9 [5]. *The set $\{\Gamma_\lambda \mid \lambda \vdash n\}$ is the set of primitive minimal elements of $Z(\mathcal{H})^+$.*

Also we have:

Lemma 4.10.

(1) $\Gamma_{(n)} = \eta_{(n)}$.
(2) If $r\tilde{T}_{w_\lambda} \leqslant h \in Z(\mathcal{H})^+$ then $r\Gamma_\lambda \leqslant h$.

Proof. (1) is immediate from Theorem 4.3(3) and the characterization of Theorem 4.8. Part (2) is [5, Corollary 4.6]. □

5. An inner product on the Hecke algebra

The standard trace function τ on $\mathcal{H}$ is defined by $\tau(\tilde{T}_w) = 1$ if $w = 1$ and 0 otherwise. A generalization of this trace was defined in [7] as follows. Fix an element $h = \sum_{w \in S_n} r_w \tilde{T}_w \in Z(\mathcal{H})$, with $r_w \in \mathbb{Z}[\xi]$. For $w \in S_n$, set $\mathfrak{h}(\tilde{T}_w) := r_w$, and extend linearly to all of $\mathcal{H}$. If $h = \tilde{T}_1$ then this is simply τ. In this section we introduce a more flexible alternative formulation of this map as an inner product.

Define a map $\langle,\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{Z}[\xi]$ by setting

$$\left\langle \sum_{w \in S_n} r_w \tilde{T}_w, \sum_{w \in S_n} r'_w \tilde{T}_w \right\rangle = \sum_{w \in S_n} r_w r'_w. \tag{5.1}$$

If $h \in Z(\mathcal{H})$, then (5.1) defines the map $\mathfrak{h}$, since $\langle h, \tilde{T}_w \rangle$ gives the coefficient of $\tilde{T}_w$ in h. This map and Proposition 5.1 were developed during discussions between the first author and Leonard Scott in 1999.

Proposition 5.1. Let $u, v, w \in S_n$. The map defined in (5.1) is an inner product on $\mathcal{H}$ satisfying

$$\langle \tilde{T}_u \tilde{T}_v \tilde{T}_{u^{-1}}, \tilde{T}_w \rangle = \langle \tilde{T}_v, \tilde{T}_{u^{-1}} \tilde{T}_w \tilde{T}_u \rangle. \tag{5.2}$$

Proof. The properties of an inner product are easily verified from the definition. We prove (5.2) by induction on the length of u.

If $\ell(u) = 1$, then we may set $u = s \in S$, and reduce the problem to considering v and w in the same $\langle s \rangle$–$\langle s \rangle$ double coset of S_n (if they are in different double cosets then both sides of (5.2) will be zero and the statement holds). Further, if $v = w$ and $u = s$ then the statement holds by the symmetry of the inner product. So to prove the result for $\ell(u) = 1$ we need to check the cases where $v \neq w$ in the same $\langle s \rangle$–$\langle s \rangle$ double coset.

There are two cases for such a double coset $\langle s \rangle d \langle s \rangle$ in the symmetric group: either $ds = sd$ or $ds \neq sd$. If $ds = sd$ then the double coset consists of only $\{d, ds\}$ and there is just one case to check:

$$\langle \tilde{T}_d, \tilde{T}_s \tilde{T}_{ds} \tilde{T}_s \rangle = \langle \tilde{T}_d, \tilde{T}_s(\tilde{T}_d + \xi \tilde{T}_{ds}) \rangle = \langle \tilde{T}_d, \tilde{T}_{sd} + \xi \tilde{T}_d + \xi^2 \tilde{T}_{ds} \rangle$$
$$= \xi = \langle \tilde{T}_d + \xi \tilde{T}_{ds}, \tilde{T}_{ds} \rangle = \langle \tilde{T}_s \tilde{T}_d \tilde{T}_s, \tilde{T}_{ds} \rangle.$$

When $ds \neq sd$ the double coset has four elements $\{d, ds, sd, sds\}$, and so there are $\binom{4}{2} = 6$ cases:

$$\langle \tilde{T}_d, \tilde{T}_s \tilde{T}_{ds} \tilde{T}_s \rangle = \langle \tilde{T}_d, \tilde{T}_{sd} + \xi \tilde{T}_{sds} \rangle = 0 = \langle \tilde{T}_{sds}, \tilde{T}_{ds} \rangle = \langle \tilde{T}_s \tilde{T}_d \tilde{T}_s, \tilde{T}_{ds} \rangle,$$

$$\langle \tilde{T}_d, \tilde{T}_s \tilde{T}_{sds} \tilde{T}_s \rangle = \langle \tilde{T}_d, \tilde{T}_d + \xi \tilde{T}_{ds} + \xi \tilde{T}_{sd} + \xi^2 \tilde{T}_{sds} \rangle = 1 = \langle \tilde{T}_s \tilde{T}_d \tilde{T}_s, \tilde{T}_{sds} \rangle,$$

$$\langle \tilde{T}_{ds}, \tilde{T}_s \tilde{T}_{sd} \tilde{T}_s \rangle = \langle \tilde{T}_{ds}, \tilde{T}_{ds} + \xi \tilde{T}_{sds} \rangle = 1 = \langle \tilde{T}_s \tilde{T}_{ds} \tilde{T}_s, \tilde{T}_{sd} \rangle,$$

$$\langle \tilde{T}_{ds}, \tilde{T}_s \tilde{T}_{sds} \tilde{T}_s \rangle = \xi = \langle \tilde{T}_s \tilde{T}_{ds} \tilde{T}_s, \tilde{T}_{sds} \rangle.$$

The cases

$$\langle \tilde{T}_d, \tilde{T}_s \tilde{T}_{sd} \tilde{T}_s \rangle = \langle \tilde{T}_s \tilde{T}_d \tilde{T}_s, \tilde{T}_{sd} \rangle \quad \text{and} \quad \langle \tilde{T}_{sd}, \tilde{T}_s \tilde{T}_{sds} \tilde{T}_s \rangle = \langle \tilde{T}_s \tilde{T}_{sd} \tilde{T}_s, \tilde{T}_{sds} \rangle$$

are symmetric to cases listed above.

Now let $u = u_1 s$ for some $u_1 \in S_n$ and $s \in S$ with $\ell(u_1 s) = \ell(u_1) + 1$. We have

$$\langle \tilde{T}_u \tilde{T}_v \tilde{T}_{u^{-1}}, \tilde{T}_w \rangle$$
$$= \langle \tilde{T}_{u_1} (\tilde{T}_s \tilde{T}_v \tilde{T}_s) \tilde{T}_{u_1^{-1}}, \tilde{T}_w \rangle$$
$$= \langle \tilde{T}_s \tilde{T}_v \tilde{T}_s, \tilde{T}_{u_1^{-1}} \tilde{T}_w \tilde{T}_{u_1} \rangle \quad \text{(by linearity of the inner product, and by induction)}$$
$$= \langle \tilde{T}_v, \tilde{T}_s (\tilde{T}_{u_1^{-1}} \tilde{T}_w \tilde{T}_{u_1}) \tilde{T}_s \rangle \quad \text{(for the same reasons)}$$
$$= \langle \tilde{T}_v, \tilde{T}_u^{-1} \tilde{T}_w \tilde{T}_u \rangle,$$

which completes the proof. $\square$

Corollary 5.2. *Let* $\lambda, \mu \vDash n$. *Then*

$$\langle \mathcal{N}_{S_n, 1}(\tilde{T}_{w_\lambda}), \tilde{T}_{w_\mu} \rangle = \langle \mathcal{N}_{S_n, 1}(\tilde{T}_{w_\mu}), \tilde{T}_{w_\lambda} \rangle.$$

Proof. Immediate from 5.1, the symmetry of the inner product, and the fact that the norm from the identity sums over the entire group. $\square$

The following lemma was stated originally in terms of the function $\mathfrak{h}$ for a fixed $h \in Z(\mathcal{H})$. The statement below is a direct translation of that result.

Lemma 5.3 [7, Lemma (3.5)]. *If* $h \in Z(\mathcal{H})$ *then* $\langle h, \tilde{T}_w \tilde{T}_v \rangle = \langle h, \tilde{T}_v \tilde{T}_w \rangle$.

A key property of the inner product in our context is the following:

Lemma 5.4. *Let* $\lambda \vdash n$. *The coefficient of* Γ_λ *in* $h \in Z(\mathcal{H})$ *is* $\langle h, \tilde{T}_{w_\lambda} \rangle$.

Proof. Immediate from Lemma 4.10(2). $\square$

6. The norm of the identity in terms of the class elements

The goal of this section is to prove Theorem 6.3, which gives the coefficient of a class element in the norm of the identity. This result is required for the proof of Theorem 7.4. Eventually, Theorem 6.3 is subsumed in the statement of Theorem 9.2.

Lemma 6.1. *For* $1 \leqslant i \leqslant n - 1$,

$$\langle \Gamma_{(n)}, \tilde{T}_{s_1 s_2 \ldots s_n s_i \ldots s_1} \rangle = \xi \langle \Gamma_{(n)}, \tilde{T}_{s_1 s_2 \ldots s_n s_{i-1} \ldots s_1} \rangle.$$

Proof. Note that $s_1 s_2 \ldots s_n s_{i-1} \ldots s_1$ and $s_1 s_2 \ldots s_n s_i \ldots s_2$ are in the same conjugacy class and are conjugate by some $x \in S_n$ satisfying the Geck–Pfeiffer property II (Theorem 1.1(ii) of [8]), and so the coefficients of their corresponding elements in any central element of the Hecke algebra are equal (using Lemma 5.3). In particular, $\langle \Gamma_{(n)}, \tilde{T}_{s_1 s_2 \ldots s_n s_{i-1} \ldots s_1} \rangle = \langle \Gamma_{(n)}, \tilde{T}_{s_1 s_2 \ldots s_n s_i \ldots s_2} \rangle$. Using Lemma 5.3 again, we have that

$$
\begin{aligned}
& \langle \Gamma_{(n)}, \tilde{T}_{s_1 s_2 \ldots s_n s_i \ldots s_1} \rangle \\
&= \langle \Gamma_{(n)}, \tilde{T}_{s_1} \tilde{T}_{s_1 s_2 \ldots s_n s_i \ldots s_2} \rangle \\
&= \langle \Gamma_{(n)}, (\tilde{T}_1 + \xi \tilde{T}_{s_1}) \tilde{T}_{s_2 \ldots s_n s_i \ldots s_2} \rangle \\
&= \langle \Gamma_{(n)}, \tilde{T}_{s_2 \ldots s_n s_i \ldots s_2} + \xi \tilde{T}_{s_1 s_2 \ldots s_n s_i \ldots s_2} \rangle \\
&= \langle \Gamma_{(n)}, \xi \tilde{T}_{s_1 s_2 \ldots s_n s_i \ldots s_2} \rangle \quad \text{(by linearity, Theorem 4.3(2) and Lemma 4.10(1))} \\
&= \langle \Gamma_{(n)}, \xi \tilde{T}_{s_1 s_2 \ldots s_n s_{i-1} \ldots s_1} \rangle,
\end{aligned}
$$

which proves the lemma. $\square$

Lemma 6.2. *For* $n \geqslant 2$, *the coefficient of* $\tilde{T}_{s_1 s_2 \ldots s_n \ldots s_2 s_1}$ *in* $\Gamma_{(n)}$ *is* ξ^{n-2}.

Proof. This follows from repeated application of Lemma 6.1. $\square$

Theorem 6.3. *The coefficient of* Γ_λ *in* $\mathcal{N}_{S_n, 1}(\tilde{T}_1)$ *is* $[S_n : S_\lambda] \xi^{l_\lambda}$. *Equivalently,*

$$\langle \mathcal{N}_{S_n, 1}(\tilde{T}_1), \tilde{T}_{w_\lambda} \rangle = [S_n : S_\lambda] \xi^{l_\lambda}$$

where w_λ *is as defined in Section* 1.2.

Proof. Firstly note that the two statements in the theorem are in fact equivalent by Lemma 5.4.

Secondly note that the symmetry of the norm from the trivial subgroup with respect to the inner product (Corollary 5.2) means that

$$\langle \mathcal{N}_{S_n, 1}(\tilde{T}_1), \tilde{T}_{w_\lambda} \rangle = \langle \mathcal{N}_{S_n, 1}(\tilde{T}_{w_\lambda}), \tilde{T}_1 \rangle.$$

Thus the problem reduces to identifying the coefficient of $\tilde{T}_1$ in the norm of $\tilde{T}_{w_\lambda}$ from the trivial subgroup to S_n.

The transitivity of the norm implies

$$
\begin{aligned}
\mathcal{N}_{S_n,1}\big(\tilde{T}_{w_\lambda}\big) &= \mathcal{N}_{S_n,S_\lambda}\big(\mathcal{N}_{S_\lambda,S_{\lambda-1}}\big(\mathcal{N}_{S_{\lambda-1},1}\big(\tilde{T}_{w_\lambda}\big)\big)\big) \\
&= \mathcal{N}_{S_n,S_\lambda}\big(\mathcal{N}_{S_\lambda,S_{\lambda-1}}(\eta_\lambda)\big) \\
&= \mathcal{N}_{S_n,S_\lambda}\big(\eta_\lambda \mathcal{N}_{S_\lambda,S_{\lambda-1}}\big(\tilde{T}_1\big)\big).
\end{aligned}
\tag{6.1}
$$

The result therefore depends on finding the coefficient of $\tilde{T}_1$ in expression (6.1). Now (6.1) is a linear combination of terms of the form $\tilde{T}_{d^{-1}}\tilde{T}_w \tilde{T}_d$ where d is a distinguished right coset representative of S_λ in S_n, and where $\tilde{T}_w$ occurs in $\eta_\lambda \mathcal{N}_{S_\lambda,S_{\lambda-1}}(\tilde{T}_1)$ (and is therefore an element of $\mathcal{H}_\lambda$). Since d is a coset representative, $\tilde{T}_1$ occurs in $\tilde{T}_{d^{-1}}\tilde{T}_w \tilde{T}_d$ only if $w = 1$ (by Theorem 3.2).

Similarly, by Theorem 3.2, it is straightforward to see that $\tilde{T}_1$ occurs in $\eta_\lambda \mathcal{N}_{S_\lambda,S_{\lambda-1}}(\tilde{T}_1)$ only when $\tilde{T}_{w^{-1}}$ occurs in $\mathcal{N}_{S_\lambda,S_{\lambda-1}}(\tilde{T}_1)$ for some $\tilde{T}_w$ occurring in η_λ.

The norm $\mathcal{N}_{S_\lambda,S_{\lambda-1}}(\tilde{T}_1)$ splits into commuting factors along the components of λ. That is, if $\lambda = (\lambda_1, \ldots, \lambda_r)$ then

$$
\mathcal{N}_{S_\lambda,S_{\lambda-1}}\big(\tilde{T}_1\big) = \prod_{i=1}^{r} \mathcal{N}_{S_{\lambda_i},S_{\lambda_i-1}}\big(\tilde{T}_1\big).
\tag{6.2}
$$

The non-trivial right coset representatives of each S_{λ_i-1} in S_{λ_i} are of the form $s_{\lambda_1+\cdots+\lambda_{i-1}-1}$ $\cdots s_{\lambda_1+\cdots+\lambda_{i-1}+\lambda_i+j}$ for $1 \leqslant j \leqslant \lambda_i - 1$. Thus each factor in (6.2) is a sum of products of the form

$$
\tilde{T}_{s_{x+j}\ldots s_{x+\lambda_i-1}} \tilde{T}_{s_{x+\lambda_i-1}\ldots s_{x+j}}
\tag{6.3}
$$

where $x = \lambda_1 + \cdots + \lambda_{i-1}$. By Lemma 3.4, each non-identity term in the expansion of (6.3) is of the form $\xi \tilde{T}_w$ where w is a transposition. Thus the non-identity terms in $\mathcal{N}_{S_\lambda,S_{\lambda-1}}(\tilde{T}_1)$ are products of terms of the form $\xi \tilde{T}_w$ where w is a transposition in S_λ. In addition, if w is the longest transposition in S_λ, then ξ^t is the coefficient of $\tilde{T}_w$ where $t = |\{\lambda_i \geqslant 2\}|$, by (6.2).

Therefore, $\eta_\lambda \mathcal{N}_{S_\lambda,S_{\lambda-1}}(\tilde{T}_1)$ contains $\tilde{T}_1$ only when η_λ contains a transposition. By Lemma 4.3(2), η_λ contains only the longest transposition in S_λ, and its coefficient by Lemma 6.2 is $\prod_{i=1}^{t} \xi^{\lambda_i-2} = \xi^{\lambda_1+\cdots+\lambda_t-2t}$. Hence, the coefficient of $\tilde{T}_1$ in $\eta_\lambda \mathcal{N}_{S_\lambda,S_{\lambda-1}}(\tilde{T}_1)$ is $\xi^{\lambda_1+\cdots+\lambda_t-2t}\xi^t = \xi^{l_\lambda}$.

So $\tilde{T}_1$ occurs in $\mathcal{N}_{S_n,1}(\tilde{T}_{w_\lambda})$ for each coset representative of S_λ in S_n. The statement follows. $\square$

7. A partial order on the norm basis and the coefficient of the Coxeter class

In this section we prove that the elements of the norm basis ordered by refinement of partitions are also ordered by the Hecke algebra order defined in Section 1.3. While this result might be of independent interest, our chief use for this fact in the present paper is to obtain explicitly the coefficient of the Coxeter class element in a given norm basis element (see Theorem 7.4).

Recall that the set $B := \{b_\alpha \mid \alpha \vdash n\}$ is the $\mathbb{Q}[\xi]$-basis for the centre $Z(\mathcal{H})$ defined in Section 4.1.

Lemma 7.1. *For any integers n and k with $n - k \leqslant k < n$, $\xi b_{(n)} < b_{(k,n-k)}$.*

Proof. Using the fact that the double coset representative d_{n-k} as defined in Section 2 is also a right coset representative of $S_{(k,n-k)}$ in S_n, we have

$$
\begin{aligned}
b_{(k,n-k)} &= \mathcal{N}_{S_n, S_{(k,n-k)}}(\eta_{(k,n-k)}) \\
&\geqslant \tilde{T}_{d_{n-k}} \tilde{T}_{s_1 \ldots s_{k-1} s_{k+1} \ldots s_{n-1}} \tilde{T}_{d_{n-k}} \\
&= \tilde{T}_{s_1 \ldots s_{k-1} s_{k+1} \ldots s_{n-1}} \tilde{T}_{d_{n-k}}^2 \quad \text{(since } d_{n-k} \text{ commutes with } s_1 \ldots s_{k-1} s_{k+1} \ldots s_{n-1}) \\
&> \tilde{T}_{s_1 \ldots s_{k-1} s_{k+1} \ldots s_{n-1}} \left(\xi \tilde{T}_{s_k}\right) \quad \text{(since } \xi \tilde{T}_{s_k} \leqslant \tilde{T}_{d_{n-k}}^2) \\
&= \xi \tilde{T}_{s_1 \ldots s_{k-1} s_{k+1} \ldots s_{n-1}} \tilde{T}_{s_k}.
\end{aligned}
$$

Since $s_1 \ldots s_{k-1} s_{k+1} \ldots s_{n-1} s_k$ is a minimal element of the Coxeter class $C_{(n)}$, and since $b_{(k,n-k)} \in Z(\mathcal{H})^+$ [5, Proposition (5.3)(ii)], it follows that $\xi \Gamma_{(n)} \leqslant b_{(k,n-k)}$ by [5, Corollary (4.6)]. Since $b_{(n)} = \Gamma_{(n)}$, and since $\tilde{T}_{w_{(k,n-k)}}$ occurs in $b_{(k,n-k)}$ but not in $b_{(n)}$, the lemma follows. $\square$

Theorem 7.2. *Let $\lambda, \mu \vDash n$. If $\lambda < \mu$ in the refinement order, then $\xi^{l_\mu - l_\lambda} b_\mu < b_\lambda$.*

Proof. We first prove the theorem when $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\lambda_1, \ldots, \lambda_{r-2}, \lambda_{r-1} + \lambda_r)$. In this case we have

$$
\begin{aligned}
b_\lambda &= \mathcal{N}_{S_n, S_\lambda}(\eta_\lambda) \\
&= \mathcal{N}_{S_n, S_\mu}\left(\mathcal{N}_{S_\mu, S_\lambda}(\eta_\lambda)\right) \\
&= \mathcal{N}_{S_n, S_\mu}\left(\left(\prod_{i=1}^{r-2} \mathcal{N}_{S_{\lambda_i}, S_{\lambda_i}}(\eta_{\lambda_i})\right) \mathcal{N}_{S_{\lambda_{r-1}+\lambda_r}, S_{(\lambda_{r-1}, \lambda_r)}}(\eta_{(\lambda_{r-1}, \lambda_r)})\right) \\
&> \mathcal{N}_{S_n, S_\mu}\left(\left(\prod_{i=1}^{r-2} \mathcal{N}_{S_{\lambda_i}, S_{\lambda_i}}(\eta_{\lambda_i})\right) \xi \mathcal{N}_{S_{\lambda_{r-1}+\lambda_r}, S_{\lambda_{r-1}+\lambda_r}}(\eta_{(\lambda_{r-1}+\lambda_r)})\right) \quad \text{(by Lemma 7.1)} \\
&= \xi \mathcal{N}_{S_n, S_\mu}\left(\mathcal{N}_{S_\mu, S_\mu}(\eta_\mu)\right) = \xi \mathcal{N}_{S_n, S_\mu}(\eta_\mu) = \xi b_\mu.
\end{aligned}
$$

Similar arguments demonstrate the validity of the theorem when $\mu = (\lambda_1, \lambda_2, \ldots, \lambda_{i-1},$ $\lambda_i + \lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_r)$ with $1 \leqslant i \leqslant r - 2$. The extension to arbitrary $\lambda < \mu$ is then immediate. $\quad\square$

Corollary 7.3. *Write* $b_\alpha = \sum_{\lambda \vdash n} r_{\alpha,\lambda} \Gamma_\lambda$ *for* $\alpha \vdash n$ *and* $r_{\alpha,\lambda} \in \mathbb{Z}[\xi]$. *Then* $\alpha < \beta$ *implies* $\xi r_{\beta,\lambda} \leqslant r_{\alpha,\lambda}$ *for any* $\lambda \vdash n$.

Proof. The partial order of Theorem 7.2 implies that for an arbitrary basis element $\tilde{T}_w$ of $\mathcal{H}$, if $\lambda < \mu$ then the coefficients $r_{w,\lambda}$ and $r_{w,\mu}$ of $\tilde{T}_w$ in b_λ and b_μ respectively satisfy the relation $\xi r_{w,\mu} \leqslant r_{w,\lambda}$. In particular, this is the case when w is a shortest element of a conjugacy class. It follows that for any sequence of partitions totally ordered by refinement between (1^n) and (n), the coefficients of any given class element also satisfy the inequality by Lemma 5.4. $\quad\square$

Theorem 7.4. *The coefficient of the Coxeter class element* $\Gamma_{(n)}$ *in* b_α *is* ξ^{n-1-l_α}.

Proof. We have from Lemma 4.10(1) that the coefficient of $\Gamma_{(n)}$ in $b_{(n)}$ is 1. We also have that the coefficient of $\Gamma_{(n)}$ in $b_{(1^n)}$ is ξ^{n-1} by Theorem 6.3. Clearly both of these coefficients satisfy the theorem statement. Given any sequence of partitions α_i totally ordered by refinement between (1^n) and (n):

$$(1^n) = \alpha_0 < \cdots < \alpha_{n-1} = (n),$$

Corollary 7.3 gives the following relations, where r_i is the coefficient of $\Gamma_{(n)}$ in b_{α_i}:

$$\xi^{n-1} = r_0 \geqslant \xi r_1 \geqslant \cdots \geqslant \xi^i r_i \geqslant \cdots \geqslant \xi^{n-1} r_{n-1} = \xi^{n-1}.$$

It follows that all the terms in the above order are equal, and $\xi^i r_i = \xi^{n-1}$, so that $r_i = \xi^{n-1-i}$. Since length increases by one with each step of the refinement order, $i = l_{\alpha_i}$. The result follows since every partition is part of such a full sequence of partitions between (1^n) and (n) ordered by refinement with length increasing by one. $\quad\square$

Remark. Applying the map from $\mathcal{H} \to \mathcal{H}_q$ (as described in the Remark in Section 1.3) to the expansion of b_α, the coefficient of the Coxeter class element in Theorem 7.4 in terms of the $\mathbb{Z}[q, q^{-1}]$-basis for $\mathcal{H}_q$ is $(q^{-1}(q-1))^{n-1-l_\alpha}$. Note that under the map $\mathcal{H} \to \mathcal{H}_q$, $\Gamma_{(n)} \mapsto q^{-(n-1)/2} \Gamma_{(n),q}$ and $b_\alpha \mapsto q^{-l_\alpha/2} b_{\alpha,q}$, where $\Gamma_{(n),q}$ and $b_{\alpha,q}$ are the corresponding $\mathcal{H}_q$ versions. This is an example of a result whose statement over $\mathbb{Z}[q, q^{-1}]$ is independent of the partial order on $\mathcal{H}^+$—which makes sense only over $\mathbb{Z}[\xi]$—but whose proof is made possible by treating the Hecke algebra as a $\mathbb{Z}[\xi]$-module and using the partial order.

8. Projections of norms onto maximal parabolic subalgebras

The main result of this section is a projection formula for b_α (a norm basis element) onto a maximal parabolic subgroup. Results in this section are from [12] but have not appeared in the literature.

Let $B = \{b_\alpha \mid \alpha \vdash n\}$ (see Section 4). If $\theta = (\theta_1, \ldots, \theta_t)$ is a multipartition of n, write $\eta_\theta := \eta_{\theta_1} \ldots \eta_{\theta_t}$.

For $\lambda \vDash n$, define the projection $\pi_\lambda : \mathcal{H}_n \to \mathcal{H}_\lambda$ by setting

$$\pi_\lambda(\tilde{T}_w) = \begin{cases} \tilde{T}_w, & \text{if } w \in S_\lambda, \\ 0, & \text{otherwise,} \end{cases}$$

and extending to $\mathcal{H}_n$ linearly.

Theorem 8.1 [12]. *Let $b_\alpha \in B$ and write $\pi := \pi_{(k,n-k)}$. Then*

$$\pi(b_\alpha) = \sum_{(\mu,\nu) \in \Lambda_{(k,n-k)}(\alpha)} z_{\mu,\nu} \mathcal{N}_{S_{(k,n-k)}, S_{(\mu,\nu)}}(\eta_{(\mu,\nu)}), \tag{8.1}$$

with the $z_{\mu,\nu}$ non-negative integers.

Proof. The proof is by induction on n. The theorem is easily verified for $n = 2$, so assume it in all cases less than n.

For $(\mu, \nu) \in \Lambda_{(k,n-k)}(\alpha)$ we have

$$\begin{aligned} b_\alpha &= \mathcal{N}_{S_n, S_\alpha}(\eta_\alpha) \\ &= \mathcal{N}_{S_n, S_{(k,n-k)}}\big(\mathcal{N}_{S_{(k,n-k)}, S_{(\mu,\nu)}}(\eta_{(\mu,\nu)})\big) \quad \text{(by Theorem 4.4 and Proposition 4.2)} \\ &= \sum_{m=0}^{n-k} \mathcal{N}_{S_{(k,n-k)}, P_m}\big(\tilde{T}_{d_m} \mathcal{N}_{S_{(k,n-k)}, S_{(\mu,\nu)}}(\eta_{(\mu,\nu)}) \tilde{T}_{d_m}\big) \quad \text{(by Theorem 4.5)} \\ &= \sum_{m=0}^{n-k} \mathcal{N}_{S_{(k,n-k)}, P_m}\big(\tilde{T}_{d_m} \mathcal{N}_{S_k, S_\mu}(\eta_\mu) \mathcal{N}_{S_{n-k}, S_\nu}(\eta_\nu) \tilde{T}_{d_m}\big) \quad \text{(by Proposition 4.6).} \end{aligned}$$

Consider the general term in this last sum. For this term, write $\mathcal{N}_{S_k, S_\mu}(\eta_\mu)$ as $A_k + B_k$ where A_k is the projection of $\mathcal{N}_{S_k, S_\mu}(\eta_\mu)$ onto $\mathcal{H}_{(k-m,m)}$ in $\mathcal{H}_k$. Similarly, write $\mathcal{N}_{S_{n-k}, S_\nu}(\eta_\nu)$ as $A_{n-k} + B_{n-k}$ where A_{n-k} is the projection of $\mathcal{N}_{S_{n-k}, S_\nu}(\eta_\nu)$ onto $\mathcal{H}_{(m,n-k-m)}$ in $\mathcal{H}_{n-k}$. Thus

$$\mathcal{N}_{S_k, S_\mu}(\eta_\mu) \mathcal{N}_{S_{n-k}, S_\nu}(\eta_\nu) = A_k A_{n-k} + A_k B_{n-k} + B_k A_{n-k} + B_k B_{n-k}.$$

By induction,

$$A_k = \sum_{(\mu_1,\mu_2) \in \Lambda_{(k-m,m)}(\mu)} z_{\mu_1,\mu_2} \mathcal{N}_{S_{(k-m,m)}, S_{(\mu_1,\mu_2)}}(\eta_{(\mu_1,\mu_2)}),$$

and

$$A_{n-k} = \sum_{(\nu_1,\nu_2) \in \Lambda_{(m,n-k-m)}(\nu)} z_{\nu_1,\nu_2} \mathcal{N}_{S_{(m,n-k-m)}, S_{(\nu_1,\nu_2)}}(\eta_{(\nu_1,\nu_2)}),$$

with the z_{μ_1,μ_2} and z_{ν_1,ν_2} non-negative integers. We abbreviate the $(k-m, m, m, n-k)$-multipartition $(\mu_1, \mu_2, \nu_1, \nu_2)$ of α by $\bar{\alpha}$. Then

$$A_k A_{n-k} = \sum_{\substack{(\mu_1,\mu_2)\in\Lambda_{(k-m,m)}(\mu)\\(\nu_1,\nu_2)\in\Lambda_{(m,n-k-m)}(\nu)}} z_{\mu_1,\mu_2} z_{\nu_1,\nu_2} \mathcal{N}_{P_m, S_{\bar{\alpha}}}(\eta_{\bar{\alpha}}) \tag{8.2}$$

by splicing the norms back together utilizing Proposition 4.6.

Each of the terms in (8.2) is then multiplied by $\tilde{T}_{d_m}$ on the left and right. Write

$$\mathcal{N}_{P_m, S_{\bar{\alpha}}}(\eta_{\bar{\alpha}}) = \sum_{r\in R} \tilde{T}_{r^{-1}} \eta_{\bar{\alpha}} \tilde{T}_r$$

where R is the set of distinguished right coset representatives for $S_{\bar{\alpha}}$ in P_m.

Because d_m is a distinguished double coset representative of $S_{(k,n-k)}$ in S_n, we have $\ell(rd_m) = \ell(d_m r^{-1}) = \ell(d_m) + \ell(r)$ for any $r \in R$. Then, since $\hat{r} = d_m r d_m$ is a distinguished right coset representative for $S_{\bar{\alpha}}^{d_m}$ in P_m (Corollary 2.5), it follows that:

$$\tilde{T}_{d_m} \tilde{T}_{r^{-1}} \eta_{\bar{\alpha}} \tilde{T}_r \tilde{T}_{d_m} = \tilde{T}_{d_m r^{-1}} \eta_{\bar{\alpha}} \tilde{T}_{rd_m} = \tilde{T}_{\hat{r}^{-1} d_m} \eta_{\bar{\alpha}} \tilde{T}_{d_m \hat{r}} = \tilde{T}_{\hat{r}^{-1}} \tilde{T}_{d_m} \eta_{\bar{\alpha}} \tilde{T}_{d_m} \tilde{T}_{\hat{r}}.$$

Since $\eta_{\bar{\alpha}} \in \mathcal{H}_{\bar{\alpha}}$ and $S_{\bar{\alpha}} \leqslant P_m$, Corollary 3.6 implies that

$$\tilde{T}_{d_m} \eta_{\bar{\alpha}} \tilde{T}_{d_m} = \eta_{\bar{\alpha}^{d_m}} + \sum_{w\in S_n} f_w \tilde{T}_w, \quad \text{where } \#(w) \geqslant 1 \text{ if } f_w \neq 0.$$

Since $\#(\hat{r}) = 0$, we have from Corollary 3.3(2) that

$$\tilde{T}_{\hat{r}^{-1}} \tilde{T}_{d_m} \eta_{\bar{\alpha}} \tilde{T}_{d_m} \tilde{T}_{\hat{r}} = \tilde{T}_{\hat{r}^{-1}} \eta_{\bar{\alpha}^{d_m}} \tilde{T}_{\hat{r}} + \sum_{w\in S_n} g_w \tilde{T}_w, \quad \text{where } \#(w) \geqslant 1 \text{ if } g_w \neq 0,$$

and so,

$$\tilde{T}_{d_m} \mathcal{N}_{P_m, S_{\bar{\alpha}}}(\eta_{\bar{\alpha}}) \tilde{T}_{d_m} = \mathcal{N}_{P_m, S_{\bar{\alpha}}^{d_m}}(\eta_{\bar{\alpha}^{d_m}}) + \sum_{w\in S_n} h_w \tilde{T}_w, \quad \text{where } \#(w) \geqslant 1 \text{ if } h_w \neq 0.$$

Thus, applying Theorem 4.4(1) to $\mathcal{H}_{(k,n-k)}$,

$$\pi\left(\mathcal{N}_{S_{(k,n-k)}, P_m}\left(\tilde{T}_{d_m} A_k A_{n-k} \tilde{T}_{d_m}\right)\right)$$

$$= \sum_{\substack{(\mu_1,\mu_2)\in\Lambda_{(k-m,m)}(\mu)\\(\nu_1,\nu_2)\in\Lambda_{(m,n-k-m)}(\nu)}} z_{\mu_1,\mu_2} z_{\nu_1,\nu_2} \mathcal{N}_{S_{(k,n-k)}, S_{\bar{\alpha}}^{d_m}}(\eta_{\bar{\alpha}^{d_m}}). \tag{8.3}$$

Collecting common terms and re-indexing, (8.3) may be written as

$$\pi\left(\mathcal{N}_{S_{(k,n-k)},P_m}\left(\tilde{T}_{d_m}A_k A_{n-k}\tilde{T}_{d_m}\right)\right)$$

$$= \sum_{(\mu,\nu)\in\Lambda_{(k,n-k)}(\alpha)} z_{\mu,\nu}\mathcal{N}_{S_{(k,n-k)},S_{(\mu,\nu)}}(\eta_{(\mu,\nu)}),$$

with the $z_{\mu,\nu}$ non-negative integers.

We now examine the product $\tilde{T}_{d_m}B_k B_{n-k}\tilde{T}_{d_m}$. Since B_k projects to zero on $\mathcal{H}_{(k-m,m)}$, every non-zero term of B_k contains s_{k-m}. Similarly, every non-zero term of B_{n-k} contains s_{k+m}. Let $\tilde{T}_{w_k}$ be a non-zero term in B_k and $\tilde{T}_{w_{n-k}}$ a non-zero term in B_{n-k}, assuming for simplicity that the coefficients are one. Now $w_k w_{n-k} \in S_{(k,n-k)}$ with $\ell(w_k w_{n-k}) = \ell(w_k) + \ell(w_{n-k})$, and since d_m is a distinguished double coset representative of $S_{(k,n-k)}$ in S_n we have

$$\tilde{T}_{d_m}\tilde{T}_{w_k}\tilde{T}_{w_{n-k}}\tilde{T}_{d_m} = \tilde{T}_{d_m}\tilde{T}_{w_k w_{n-k}d_m}. \tag{8.4}$$

If $d_m w_k w_{n-k}d_m \in S_{(k,n-k)}$, then $d_m w_k w_{n-k}d_m \in P_m$ (by definition of P_m), which implies that $w_k w_{n-k} \in P_m^{d_m} = P_m$. This is a contradiction since w_k contains $s_{k-m} \notin P_m$ or since w_{n-k} contains $s_{k+m} \notin P_m$. Thus, Corollary 3.3(1) implies that every term in (8.4) contains s_k since $\#(d_m w_k w_{n-k}d_m) > 0$. Hence, every term of $\tilde{T}_{d_m}B_k B_{n-k}\tilde{T}_{d_m}$ contains s_k.

Similarly, every term of the products $\tilde{T}_{d_m}B_k A_{n-k}\tilde{T}_{d_m}$ and $\tilde{T}_{d_m}A_k B_{n-k}\tilde{T}_{d_m}$ contains s_k. Applying Corollary 3.3(2), we conclude that

$$\pi\left(\mathcal{N}_{S_{(k,n-k)},P_m}\left(\tilde{T}_{d_m}(B_k A_{n-k} + A_k B_{n-k} + B_k B_{n-k})\tilde{T}_{d_m}\right)\right) = 0.$$

This completes the proof of the theorem. $\quad\square$

Corollary 8.2. *For each* $(\mu,\nu) \in \Lambda_{(k,n-k)}$,

$$z_{\mu,\nu} = \left[N_{S_n}(S_{(\mu,\nu)}) : N_{S_{(k,n-k)}}(S_{(\mu,\nu)})\right].$$

Proof. We have from Theorem 8.1 that

$$\pi(b_\alpha) = \sum_{(\mu,\nu)\in\Lambda_{(k,n-k)}(\alpha)} z_{\mu,\nu}\mathcal{N}_{S_{(k,n-k)},S_{(\mu,\nu)}}(\eta_{(\mu,\nu)}).$$

By Proposition 4.7, when we specialize b_α by setting $\xi = 0$ we obtain $[N_{S_n}(S_\alpha) : S_\alpha]\underline{C_\alpha}$. When we project $\underline{C_\alpha}$ via π we get a sum of disjoint subsets of C_α, corresponding to $(k, n-k)$-multipartitions (μ,ν) of α. Thus

$$\pi(b_\alpha)|_{\xi=0} = \sum_{(\mu,\nu)\in\Lambda_{(k,n-k)}(\alpha)} \left[N_{S_n}(S_{(\mu,\nu)}) : S_{(\mu,\nu)}\right]\underline{C_{(\mu,\nu)}}.$$

On the other hand

$$\mathcal{N}_{S_{(k,n-k)},S_{(\mu,\nu)}}(\eta_{(\mu,\nu)})|_{\xi=0} = \left[N_{S_{(k,n-k)}}(S_{(\mu,\nu)}) : S_{(\mu,\nu)}\right]\underline{C_{(\mu,\nu)}}.$$

Consequently, for each $(\mu, \nu) \in \Lambda_{(k,n-k)}(\alpha)$ we have

$$\left[N_{S_n}(S_{(\mu,\nu)}) : S_{(\mu,\nu)}\right] = z_{\mu,\nu}\left[N_{S_{(k,n-k)}}(S_{(\mu,\nu)}) : S_{(\mu,\nu)}\right],$$

and the corollary follows. $\square$

9. Coefficients of class elements in norms

In this section we generalize Theorem 8.1 to arbitrary parabolic subalgebras, and use this generalization to prove Theorem 9.2—our main result—on the coefficients of class elements in norms.

Theorem 9.1. *Let* $\lambda \vDash n$ *and* $\alpha \vdash n$. *Then*

$$\pi_\lambda(b_\alpha) = \frac{|S_n|}{|S_\lambda| \cdot |C_\alpha|_{S_n}} \sum_{\theta \in \Lambda_\lambda(\alpha)} |C_\theta|_{S_\lambda} \mathcal{N}_{S_\lambda, S_\theta}(\eta_\theta). \tag{9.1}$$

Proof. If λ has one component, $\lambda = (n)$, then there is only one multipartition θ of shape λ using the components of α, namely $\theta = (\alpha)$ itself, and the statement is trivial. If $\lambda = (k, n-k)$ then we can apply Theorem 8.1 and Corollary 8.2 to obtain

$$\pi_\lambda(b_\alpha) = \sum_{\theta \in \Lambda_\lambda(\alpha)} \left[N_{S_n}(S_\theta) : N_{S_\lambda}(S_\theta)\right]\mathcal{N}_{S_\lambda, S_\theta}(\eta_\theta).$$

An elementary calculation gives that

$$\left[N_{S_n}(S_\theta) : N_{S_\lambda}(S_\theta)\right] = \frac{|S_n| \cdot |C_\theta|_{S_\lambda}}{|S_\lambda| \cdot |C_\alpha|_{S_n}},$$

and the statement for the case in which λ has two components follows.

Now suppose inductively for some fixed $\lambda = (\lambda_1, \ldots, \lambda_r) \vDash n$, with $r > 2$, that (9.1) holds for all compositions of n with fewer components than λ. In particular, we assume that (9.1) holds for $\lambda' = (\lambda_1, \ldots, \lambda_{r-2}, \lambda'_{r-1})$ where $\lambda'_{r-1} = \lambda_{r-1} + \lambda_r$.

Write $\theta' = (\theta_1, \ldots, \theta_{r-2}, \theta'_{r-1}) \in \Lambda_{\lambda'}(\alpha)$ with $\theta_i \vdash \lambda_i$ for $1 \leqslant i \leqslant r-2$ and $\theta'_{r-1} \vdash \lambda'_{r-1}$. By our induction hypothesis we have

$$\pi_{\lambda'}(b_\alpha) = \frac{|S_n|}{|S_{\lambda'}| \cdot |C_\alpha|_{S_n}} \sum_{\theta' \in \Lambda_{\lambda'}(\alpha)} |C_{\theta'}|_{S_{\lambda'}} \mathcal{N}_{S_{\lambda'}, S_{\theta'}}(\eta_{\theta'})$$

$$= \frac{|S_n|}{|S_{(\lambda_1, \ldots, \lambda_{r-2})}| \cdot |S_{\lambda'_{r-1}}| \cdot |C_\alpha|_{S_n}} \sum_{\theta' \in \Lambda_{\lambda'}(\alpha)} \left(|C_{(\theta_1, \ldots, \theta_{r-2})}|_{S_{(\lambda_1, \ldots, \lambda_{r-2})}} \cdot |C_{\theta'_{r-1}}|_{S_{\lambda'_{r-1}}}\right.$$

$$\left. \times \mathcal{N}_{S_{(\lambda_1, \ldots, \lambda_{r-2})}, S_{(\theta_1, \ldots, \theta_{r-2})}}(\eta_{(\theta_1, \ldots, \theta_{r-2})}) \cdot \mathcal{N}_{S_{\lambda'_{r-1}}, S_{\theta'_{r-1}}}(\eta_{\theta'_{r-1}})\right). \tag{9.2}$$

Let $\pi_{\lambda''}$ be the projection from the algebra $\mathcal{H}_{\lambda'}$ onto $\mathcal{H}_\lambda$. Then $\pi_{\lambda''}$ acts as the identity on $\mathcal{H}_{(\lambda_1,\ldots,\lambda_{r-2})}$, and acts on norms in $\mathcal{H}_{\lambda'_{r-1}}$ according to Eq. (8.1) and by the inductive hypothesis according to (9.1).

Then $\pi_\lambda = \pi_{\lambda''}\pi_{\lambda'}$, and $\pi_{\lambda''}$ acts trivially on all but the last factor in each summand of (9.2), namely

$$\mathcal{N}_{S_{\lambda'_{r-1}},S_{\theta'_{r-1}}}(\eta_{\theta'_{r-1}}).$$

The image of this norm under the projection $\pi_{\lambda''}$ to the maximal parabolic subgroup $S_{(\lambda_{r-1},\lambda_r)}$ of $S_{\lambda_{r-1}+\lambda_r}$ can thus be decomposed by induction as follows:

$$
\begin{aligned}
&\pi_{\lambda''}\big(\mathcal{N}_{S_{\lambda'_{r-1}},S_{\theta'_{r-1}}}(\eta_{\theta'_{r-1}})\big) \\
&= \frac{|S_{\lambda'_{r-1}}|}{|S_{(\lambda_{r-1},\lambda_r)}| \cdot |C_{\theta'_{r-1}}|S_{\lambda'_{r-1}}} \\
&\quad \times \sum_{(\theta_{r-1},\theta_r)\in\Lambda_{(\lambda_{r-1},\lambda_r)}(\theta'_{r-1})} |C_{(\theta_{r-1},\theta_r)}|S_{(\lambda_{r-1},\lambda_r)} \cdot \mathcal{N}_{S_{(\lambda_{r-1},\lambda_r)},S_{(\theta_{r-1},\theta_r)}}(\eta_{(\theta_{r-1},\theta_r)}).
\end{aligned}
$$

Composing $\pi_{\lambda''}$ and $\pi_{\lambda'}$ to give π_λ we have (after some initial cancellations):

$$
\begin{aligned}
&\pi_\lambda(b_\alpha) \\
&= \frac{|S_n|}{|S_{(\lambda_1,\ldots,\lambda_{r-2})}| \cdot |C_\alpha|S_n} \\
&\quad \times \sum_{\theta'\in\Lambda_{\lambda'}(\alpha)} \frac{|C_{(\theta_1,\ldots,\theta_{r-2})}|S_{(\lambda_1,\ldots,\lambda_{r-2})}}{|S_{(\lambda_{r-1},\lambda_r)}|} \cdot \mathcal{N}_{S_{(\lambda_1,\ldots,\lambda_{r-2})},S_{(\theta_1,\ldots,\theta_{r-2})}}(\eta_{(\theta_1,\ldots,\theta_{r-2})}) \\
&\quad \times \sum_{(\theta_{r-1},\theta_r)\in\Lambda_{(\lambda_{r-1},\lambda_r)}(\theta'_{r-1})} |C_{(\theta_{r-1},\theta_r)}|S_{(\lambda_{r-1},\lambda_r)} \cdot \mathcal{N}_{S_{(\lambda_{r-1},\lambda_r)},S_{(\theta_{r-1},\theta_r)}}(\eta_{(\theta_{r-1},\theta_r)}) \\
&= \frac{|S_n|}{|S_{(\lambda_1,\ldots,\lambda_{r-2})}| \cdot |S_{(\lambda_{r-1},\lambda_r)}| \cdot |C_\alpha|S_n} \\
&\quad \times \sum_{\theta\in\Lambda_\lambda(\alpha)} |C_{(\theta_1,\ldots,\theta_{r-2})}|S_{(\lambda_1,\ldots,\lambda_{r-2})} \cdot |C_{(\theta_{r-1},\theta_r)}|S_{(\lambda_{r-1},\lambda_r)} \\
&\quad \times \mathcal{N}_{S_{(\lambda_1,\ldots,\lambda_{r-2})},S_{(\theta_1,\ldots,\theta_{r-2})}}(\eta_{(\theta_1,\ldots,\theta_{r-2})}) \cdot \mathcal{N}_{S_{(\lambda_{r-1},\lambda_r)},S_{(\theta_{r-1},\theta_r)}}(\eta_{(\theta_{r-1},\theta_r)}) \\
&= \frac{|S_n|}{|S_\lambda| \cdot |C_\alpha|S_n} \sum_{\theta\in\Lambda_\lambda(\alpha)} |C_\theta|S_\lambda \cdot \mathcal{N}_{S_\lambda,S_\theta}(\eta_\theta),
\end{aligned}
$$

which completes the proof. $\quad\square$

Remark. Note that when $\lambda \neq (n)$, the norms in the summation in the statement of Theorem 9.1 are only up to a proper subgroup S_λ of S_n. Therefore, since these multipartitions $\theta \in \Lambda_\lambda(\alpha)$ are not conjugate as λ-multipartitions, Proposition 4.2 does not apply and the norms cannot be pulled out of the summation.

Theorem 9.2. *Let* $b_\alpha \in B$ *and let* Γ_λ *be a class element. Then*

$$b_\alpha = \sum_{\lambda \vdash n} (1_{S_\lambda})^{S_n} (w_\alpha) \xi^{l_\lambda - l_\alpha} \Gamma_\lambda.$$

Proof. Firstly recall that the coefficient of Γ_λ in b_α is exactly $\langle b_\alpha, \tilde{T}_{w_\lambda} \rangle$ by Lemma 5.4. The critical observation is that this value is preserved under the projection $\pi_\lambda(b_\alpha)$ onto $\mathcal{H}_\lambda$, so that $\langle b_\alpha, \tilde{T}_{w_\lambda} \rangle = \langle \pi_\lambda(b_\alpha), \tilde{T}_{w_\lambda} \rangle$. By linearity of the inner product we have

$$\langle b_\alpha, \tilde{T}_{w_\lambda} \rangle = \langle \pi_\lambda(b_\alpha), \tilde{T}_{w_\lambda} \rangle$$

$$= \frac{|S_n|}{|S_\lambda| \cdot |C_\alpha|_{S_n}} \sum_{\theta \in \Lambda_\lambda(\alpha)} |C_\theta|_{S_\lambda} \langle \mathcal{N}_{S_\lambda, S_\theta}(\eta_\theta), \tilde{T}_{w_\lambda} \rangle.$$

Now regardless of which $\theta \in \Lambda_\lambda(\alpha)$ is taken, $l_\theta = l_\alpha$, and so by applying Theorem 7.4 to each component of S_λ, the coefficient of the canonical Coxeter element $\tilde{T}_{w_\lambda}$ is $\xi^{l_\lambda - l_\alpha}$ in each case. Thus $\langle \mathcal{N}_{S_\lambda, S_\theta}(\eta_\theta), \tilde{T}_{w_\lambda} \rangle = \xi^{l_\lambda - l_\alpha}$ is independent of θ, and so may be pulled out of the summation. Then,

$$\frac{|S_n|}{|S_\lambda| \cdot |C_\alpha|_{S_n}} \sum_{\theta \in \Lambda_\lambda(\alpha)} |C_\theta|_{S_\lambda} = \frac{|S_n|}{|C_\alpha|_{S_n}} \sum_{\theta \in \Lambda_\lambda(\alpha)} \frac{|C_\theta|_{S_\lambda}}{|S_\lambda|}$$

$$= |C_{S_n}(w_\alpha)| \sum_{\theta \in \Lambda_\lambda(\alpha)} \frac{1}{|C_{S_\lambda}(w_\theta)|}$$

$$= (1_{S_\lambda})^{S_n}(w_\alpha), \tag{9.3}$$

which completes the proof. $\quad\square$

Remark. Utilizing the map $\mathcal{H} \to \mathcal{H}_q$ as described in the remark in Section 1.3, this theorem can ultimately be translated into a corresponding result over $\mathbb{Z}[q, q^{-1}]$ by considering the analogous norms b_α and class elements Γ_λ over that ring, and replacing the ξ in the statement of the theorem with $q^{-1}(q - 1)$. See the remark after Theorem 7.4.

Remark. Theorem 9.2 generalizes the result for the norm of the identity $b_{(1^n)}$ in Theorem 6.3, effectively making 6.3 redundant. It also includes as a special case Theorem 7.4, which gives the coefficient of the Coxeter class element in b_α.

An immediate generalization is the following:

Corollary 9.3. *Let* $\lambda \vDash n$ *and* $\mu \leqslant \lambda$. *Then*

$$\mathcal{N}_{S_\lambda, S_\mu}(\eta_\mu) = \sum_{\theta \in \Lambda_\lambda} (1_{S_\theta})^{S_\lambda}(w_\mu)\xi^{l_\theta - l_\mu}\Gamma_\theta.$$

10. Examples

In this section, we illustrate Theorem 9.2 by explicitly giving norms and class elements in terms of the standard basis for $\mathcal{H}$ when $n = 3$; we give tables of coefficients for $3 \leqslant n \leqslant 5$ which were obtained using Theorem 9.2; and we demonstrate use of the formula to obtain a coefficient in S_{10}.

10.1. Expansions for norms and class elements when $n = 3$

Explicit expressions for norms and class elements for Hecke algebras of types S_3 and S_4 are listed in [13] (over $\mathbb{Z}[q, q^{-1}]$) and [5] respectively. We reproduce them for $n = 3$ here:

The norms:

$$\begin{aligned}
b_{(1,1,1)} &= 6\tilde{T}_1 + 3\xi\tilde{T}_{s_1} + 3\xi\tilde{T}_{s_2} + \xi^2\tilde{T}_{s_1s_2} + \xi^2\tilde{T}_{s_2s_1} + \xi(3 + \xi^2)\tilde{T}_{s_1s_2s_1} \\
&= 6\Gamma_{(1,1,1)} + 3\xi\Gamma_{(2,1)} + \xi^2\Gamma_{(3)}, \\
b_{(2,1)} &= \tilde{T}_{s_1} + \tilde{T}_{s_2} + \xi\tilde{T}_{s_1s_2} + \xi\tilde{T}_{s_2s_1} + (1 + \xi^2)\tilde{T}_{s_1s_2s_1} = \Gamma_{(2,1)} + \xi\Gamma_{(3)}, \\
b_{(3)} &= \tilde{T}_{s_1s_2} + \tilde{T}_{s_2s_1} + \xi\tilde{T}_{s_1s_2s_1} = \Gamma_{(3)},
\end{aligned}$$

where the Γ_λ are the following class elements:

$$\begin{aligned}
\Gamma_{(1,1,1)} &= \tilde{T}_1, \\
\Gamma_{(2,1)} &= \tilde{T}_{s_1} + \tilde{T}_{s_2} + \tilde{T}_{s_1s_2s_1}, \\
\Gamma_{(3)} &= \tilde{T}_{s_1s_2} + \tilde{T}_{s_2s_1} + \xi\tilde{T}_{s_1s_2s_1}.
\end{aligned}$$

10.2. Coefficient tables when $3 \leqslant n \leqslant 5$

In the tables below, the row labelled by α gives the coefficients of the class elements in b_α. That is, the entry in position (α, λ) gives the coefficient of Γ_λ in b_α.

$n = 3$	(1^3)	$(2,1)$	(3)
(1^3)	6	3ξ	ξ^2
$(2,1)$		1	ξ
(3)			1

$n = 4$	(1^4)	$(2,1^2)$	$(3,1)$	(2^2)	(4)
(1^4)	24	12ξ	$4\xi^2$	$6\xi^2$	ξ^3
$(2,1^2)$		2	2ξ	2ξ	ξ^2
$(3,1)$			1	0	ξ
(2^2)				2	ξ
(4)					1

$n = 5$	(1^5)	$(2, 1^3)$	$(3, 1^2)$	$(2^2, 1)$	$(4, 1)$	$(3, 2)$	(5)
(1^5)	120	60ξ	$20\xi^2$	$30\xi^2$	$5\xi^3$	$10\xi^3$	ξ^4
$(2, 1^3)$		6	6ξ	6ξ	$3\xi^2$	$4\xi^2$	ξ^3
$(3, 1^2)$			2	0	2ξ	ξ	ξ^2
$(2^2, 1)$				2	ξ	2ξ	ξ^2
$(4, 1)$					1	0	ξ
$(3, 2)$						2	ξ
(5)							1

10.3. A coefficient in S_{10}

Suppose we want the coefficient of $\Gamma_{(5,3,2)}$ in $b_{(3,2,2,1,1,1)}$. Using the form of the coefficients in Theorem 9.2 from the left-hand side of Eq. (9.3), this is

$$\frac{|S_{10}|\xi^{l(5,3,2)-l(3,2,2,1,1,1)}}{|S_{(5,3,2)}| \cdot |C_{(3,2,2,1,1,1)}|_{S_{10}}} \sum_{\theta \in \Lambda_{(5,3,2)}((3,2,2,1,1,1))} |C_\theta|_{S_{(5,3,2)}}.$$

We have

$$\Lambda_{(5,3,2)}\big((3, 2, 2, 1, 1, 1)\big)$$
$$= \big\{\big((3, 2), (2, 1), (1, 1)\big), \big((3, 2), (1, 1, 1), (2)\big), \big((3, 1, 1), (2, 1), (2)\big),$$
$$\big((2, 2, 1), (3), (1, 1)\big), \big((2, 1, 1, 1), (3), (2)\big)\big\},$$

and

$$|C_{((3,2),(2,1),(1,1))}|_{S_{(5,3,2)}} = \frac{5 \cdot 4 \cdot 3}{3} \cdot \frac{3 \cdot 2}{2} = 60,$$

$$|C_{((3,2),(1,1,1),(2))}|_{S_{(5,3,2)}} = \frac{5 \cdot 4 \cdot 3}{3} = 20,$$

$$|C_{((3,1,1),(2,1),(2))}|_{S_{(5,3,2)}} = \frac{5 \cdot 4 \cdot 3}{3} \cdot \frac{3 \cdot 2}{2} = 60,$$

$$|C_{((2,2,1),(3),(1,1))}|_{S_{(5,3,2)}} = \frac{(5 \cdot 4)(3 \cdot 2)}{2 \cdot 2} \cdot \frac{3 \cdot 2 \cdot 1}{3} = 60,$$

$$|C_{((2,1,1,1),(3),(2))}|_{S_{(5,3,2)}} = \frac{5 \cdot 4}{2} \cdot \frac{3 \cdot 2 \cdot 1}{3} = 20.$$

So our coefficient is

$$\frac{10!\xi^{(4+2+1)-(2+1+1)}}{5!3!2!\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 2}}(60 + 20 + 60 + 60 + 20) = 11\xi^3.$$

Acknowledgment

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